极大平面图的结构与着色理论

(4) σ -运算与 Kempe 等价类

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摘 要：设 G 是一个 k-色图，若 G 的所有 k-着色是 Kempe 等价的，则称 G 为 Kempe 图。表征色数 ≥ 3 的 Kempe 图特征是一尚待解决难题。该文对极大平面图的 Kempe 等价性进行了研究，其主要贡献是：(1)发现导致两个 4-着色是 Kempe 等价的关键子图为 2-色耳，故对 2-色耳的特征进行了深入研究；(2)引入 σ-特征图，清晰地刻画了一个图中所有 4-着色之间的关联关系，并深入研究了 σ-特征图的性质；(3)揭示了 4-色非 Kempe 极大平面图的 Kempe 等价类可分为树型，圈型和循环圈型，并指出这 3 种类型可同时存在于一个极大平面图的 4-着色集中；(4)研究了 Kempe 极大平面图特征，给出了该类图的多米诺递推构造法，以及两个 Kempe 极大平面图猜想。

关键词：Kempe 极大平面图；Kempe 变换；σ-运算；Kempe 等价类；σ-特征图；2-色耳
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Theory on Structure and Coloring of Maximal Planar Graphs

(4) σ-Operations and Kempe Equivalent Classes

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Abstract: Let G be a k-chromatic graph. G is called a Kempe graph if all k-colorings of G are Kempe equivalent. It is an unsolved and hard problem to characterize the properties of Kempe graphs with chromatic number ≥ 3. The Kempe equivalence of maximal planar graphs is addressed in this paper. Our contributions are as follows: (1) Observe and study a class of subgraphs, called 2-chromatic ears, which play a critical role in guaranteeing the Kempe equivalence between two 4-colorings; (2) Introduce and explore the properties of σ-characteristic graphs, which clearly characterize the relations of all 4-colorings of a graph; (3) Divide the Kempe equivalent classes of non-Kempe 4-chromatic maximal planar graphs into three classes, tree-type, cycle-type, and circular-cycle-type, and point out that all these three classes can exist simultaneously in the set of 4-colorings of one maximal planar graph; (4) Study the characteristics of Kempe maximal planar graphs, introduce a recursive domino method to construct such graphs, and propose two conjectures.

Key words: Kempe maximal planar graph, Kempe transformation, σ-operation, Kempe equivalent class, σ-characteristic graph, 2-chromatic ear

1 引言

无论是理论上还是应用上，平面图都是一类非常重要的图类。理论方面，著名的 4-色猜想，唯一 4-色平面图猜想，9-色猜想，及 3-色问题等[1]不仅在图论领域，乃至整个数学界都具有重大影响；从应用的角度来讲，平面图理论可直接应用于电路布线[2]，信息科学[3]等领域。

极大平面图是平面图中一类重要的图类，它的每个面的边界均为三角形，故也称为三角剖分图。由于著名的 4-色猜想的研究对象可归为极大平面图，因此，100 多年来，关于极大平面图的研究吸引了众多学者，他们围绕着 4-色猜想，相继研究了

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度序列，构造，着色特性，可遍历性，生成运算等诸多方面[4]。并且在研究过程中，提出了诸如唯一 4-色极大平面图猜想以及 9-色猜想等，它们也逐渐构成了极大平面图着色理论的核心研究目标。

在极大平面图的着色性质与结构方面，一项很重要的工作是 Kempe 变换与 Kempe 等价类。**Kempe 变换**是指在一个着色图中，将某个 2-色导出子图的某个连通分支上的颜色互换，其余顶点的着色不变的一种颜色变换方法。本文所定义的σ-运算是指在 4-着色下的包含一次或多次 Kempe 变换的运算(详见第 3 节)。σ-运算是一种**导色运算**：从G中的一个 4-着色导出G的另一个 4-着色。

一个$k(\geq 2)$-色图G中的两种k-着色f与f'称为 **Kempe 等价的**，如果从f出发，通过若干次 Kempe 变换可获得f'。显然，这是一种等价关系，由此等价关系可对G中的着色进行划分。图G的基于f的 **Kempe 等价类**是指，所有与f互为 Kempe 等价的着色构成之集。特别，一个 4-色极大平面图G为 **Kempe 极大平面图**，如果G中任意两个 4-着色是 Kempe 等价的。

Kempe 变换始于 1879 年 Kempe 的工作[5]。自 Kempe 之后，首先对 Kempe 等价类进行系统研究的是 Fisk，他在 1977 年文献[6]中证明了：当一个极大平面图的顶点度数均为偶数时，其所有 4-着色构成一个 Kempe 等价类；1978 年，Meyniel 证明了：任一平面图的所有 5-着色构成一个 Kempe 等价类[7]；1981 年，Vergnas 与 Meyniel 证明了：不可收缩至K_5的任一简单图的所有 5-着色是一个 Kempe 等价类[8]；2007 年，Mohar 证明了：每个色数小于k的平面图，其所有k-着色是 Kempe 等价的[9]，并提出了下列猜想：

猜想 1 对$k\geq 3$，任一非完全k-正则图的所有k-着色是 Kempe 等价的[9]。

2015 年，Feghali 等人[10]证明了猜想 1 在k=3 时，除K_4或三角柱外，立方体图的所有 3-着色是 Kempe 等价的；我们证明了当$k=4$时，猜想 1 成立[1)]；当$k\geq 5$时，此猜想尚待解决。

2008 年，Cereceda 等人[11]对G的k-色特征图$P_k(G)$进行了研究，其中，$P_k(G)$的顶点集为G的所有k-着色构成的集合，两个k-着色相连边当且仅当它们在G中仅有一个顶点着不同颜色，并证明了：当G的色数$k\in\{2,3\}$时，$P_k(G)$不连通；当$k\geq 4$时，$P_k(G)$可能连通，也可能不连通。

类似于顶点着色，一些学者对边着色的 Kempe 变换与 Kempe 等价类进行了研究，如 Mcdonald 等人[12]，Sarah 等人[13]等。

有关 Kempe 变换与 Kempe 等价类的图算法复杂度的研究进展可参见文献[14−24]。

本系列文章意在建立极大平面图着色运算系统，该系统有两种导色运算：一种是σ-运算，另一种是τ-运算，也称为**伪边导色法**(另行文)。而在一个极大平面图G中，σ-运算一般不能从一个 4-着色导出所有 4-着色，或所需的 4-着色，但τ-运算可从$C_4^0(G)$中的一个 4-着色导出所有的 4-着色，或所需 4-着色。

本文所言之图皆指有限简单无向图。对于给定图G，分别用$V(G)$，$E(G)$，$d_G(v)$和$N_G(v)$来表示图G的顶点集，边集，顶点v的度数和顶点v的邻域 (即与顶点v相邻的所有顶点构成的集合)，可分别简记为V, E, $d(v)$, $N(v)$。图G的阶是$V(G)$中元素的个数$|V(G)|$。若$V'\subseteq V$，$E'\subseteq E$，且E'中每条边的两个端点均在V'中，则称图$H=(V',E')$是图G的一个**子图**。在子图H中，如果对于$\forall u,v\in V(H)$，u,v在G中相邻当且仅当它们在图H中相邻，则称H为G的一个由V'**导出的子图**，记为$G[V']$。对于点不交的两个图G，H，若将图G中的每个顶点与图H中的每个顶点相连边，则得到的新图称为图G与图H的**联图**，记为$G\vee H$。用K_n表示n-阶完全图。平凡图K_1与n阶圈C_n的联图$C_n\vee K_1$称作**轮图**W_n，其中C_n称为该**轮之圈**；构成K_1的顶点称为该轮的**轮心**。若$K_1=\{x\}$，有时把轮图W_n的圈C_n也用C^x来表示。

图G的一个**顶点着色**f是指对图G中的每个顶点分配一种颜色，简称为**着色**。如果G中每条边的两个端点分配不同的颜色，则称f为G的**正常着色**。若无特别声明，本文所言的顶点着色均指正常顶点着色。图G的一个**正常**k**-顶点着色**，简称为k**-顶点着色**或k**-着色**，是指从图G的顶点集V到颜色集$C(k)=\{1,2,\cdots,k\}$的一个映射f，满足对任意的$xy\in E(G)$，有$f(x)\neq f(y)$。如果在G中存在一个正常k-顶点着色，则称图G是k**-可着色的**。图G的**色数**，记作$\chi(G)$，是指满足图G为k-顶点可着色的最小数值k。若$\chi(G)=k$，则称G是k-色图。图G的每一个k-顶点着色f对应于顶点集V的一个划分$\{V_1,V_2,\cdots,V_k\}$，也记作$f=(V_1,V_2,\cdots,V_k)$，V_i为色组，表示分配到颜色i的所有顶点构成的集合，即

$$V(G)=\bigcup_{i=1}^{k}V_i,\ V_i\neq\phi,\ V_i\cap V_j=\phi,\ i\neq j,\ i,j=1,2,\cdots,k$$

其中$V_i\ (i=1,2,\cdots,k)$是G的独立集。图G中所有不同的k-着色所构成的集合用$C_k(G)$表示。用$C_k^0(G)$

[1)]LIU Xiaoqing, XU Jin, submitted to Discrete Mathematics.

表示 G 的所有由 k 个色组构成的划分的集，简称为图 G 的 k -**色组划分集**。

设 G 是一个 k -可着色的图，f 是它的一个 k -着色，$\{1,2,\cdots,k\}$ 为颜色集。用 G_{it}^{f} 表示在 f 下，G 中所有着颜色 i 与颜色 t 的顶点构成的顶点子集导出的子图，并称之为 **2-色导出子图**，其中，$i,t=1,2,\cdots,k$, $i\neq t$ 。在不致混淆的情况下，用 G_{it} 来代替 G_{it}^{f} ; G_{it} 中的分支也称为 it -**分支**。

如果一个图能够画在平面上使得它的边仅在顶点相交，则称这个图为**平面图**。平面图的这种画法称为它的一个**平面嵌入**，本文所研究的平面图均指基于它的一个平面嵌入的平面图。对于一个平面图，如果只要任何两个不相邻的顶点之间再加一条边，其平面性一定被破坏，则称该平面图为**极大平面图**。若一个平面图的每个面(包括无穷面)都由 3 条边界组成，则称该平面图为**三角剖分图**。易证，极大平面图和三角剖分图是等价的。

把图 1 中所示的图称为**漏斗**，其中度数为 1 的顶点称为**漏顶**；度数为 3 的顶点称为**漏腰**；两个度数为 2 的顶点称为**漏底**。若一个图的顶点导出子图是漏斗，则该子图称为**漏斗子图**。更详细论述见本系列文章(2)[25]。

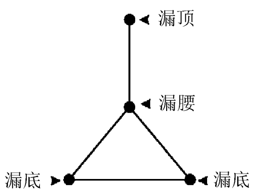


图 1 漏斗

设 G 是一个最小度 ≥ 4 的 4-色极大平面图，L 是 G 的一个漏斗子图。定义

$$f_L^4(G)=\left\{f:f\in C_4^0(G),\left|f(L)\right|=4\right\} \tag{1}$$

若 $f_L^4(G)\neq\phi$ ，则把 L 称为 G 的一个**可 4-色漏斗**，用 $L^4(G)$ 表示 G 中所有可 4-色漏斗构成之集。

本文是研究 σ -运算的首篇，发现通过 σ -运算，可从一个 4-着色导出另一个 4-着色的内在机理与一种称为 **2-色耳**的结构息息相关，于是，在第 2 节对 2-色耳展开了深入研究；第 3 节引入了 σ -特征图，清楚地刻画了基于 σ -运算的 $C_4^0(G)$ 中所有 4-着色之间的相关关系；第 4 节分析了 σ -运算的瓶颈，即在 $C_4^0(G)$ 中，有些着色之间不能通过 σ -运算相互导出；发现了极大平面图中基于 σ -运算的 Kempe 等价类共有 4 种：树型，圈型，循环圈型及 Kempe 图型；第 5 节对 Kempe 图特征进行了研究，特别，提出了 2 个关于 Kempe 图的猜想。

文中未给出的相关定义、记号与理论参见文献[25−27]。

2　2-色耳

本节先对 2-色圈给予描述与分类，在此基础上给出 2-色耳的定义与结构等。2-色耳是可连续施行 σ -运算的根源。

2.1 2-色圈相关定义

设 G 是一个 4-色极大平面图，H 是 G 的一个子图，f 是 G 的一个 4-着色。我们用 $f(H)$ 表示 H 在 f 着色下的颜色集。对于 G 中的任意圈 C ，文中分别用 $V_C^{\text{in}},V_C^{\text{out}}$ 表示 C 的内部顶点，外部顶点构成的顶点子集。

设 C 是 G 的一个偶圈，f 是 G 的一个 4-着色，若 $\left|f(C)\right|=2$ ，则称 C 是 f 的 **2-色圈**，也称 f **含 2-色圈** C , C 上的两种颜色称为**圈色**，其它两种颜色称为**非圈色**。对 f 的一个 2-色圈 C ，若 C 上存在两个顶点 u,v ，它们在 C 上不相邻，但在 G 中相邻且边 uv 在 C 内部，则称边 uv 为 2-色圈 C 的**弦**，且称 C 为 **2-色弦圈**；若 u,v 之间存在一条顶点数 ≥ 3 的路 P ，P 的内部顶点都在 C 的内部，P 上顶点的颜色均为圈色，则称 P 为 C 的**弦路**，且称 C 为 **2-色弦路圈**；如果一个 2-色圈 C 的内部不含弦及弦路，称 C 为 **2-色基本圈**。图 2(a)，图 2(b)所示的圈 C 分别为 2-色弦圈与 2-色弦路圈，其中图 2(a)中的弦为 v_1y_5 ，图 2(b)中的弦路为 $v_1v_2v_3v_4v_5$ ；图 2(a)与图 2(b)中的圈 C_1 与 C_2 均为 2-色基本圈。

基于 4-着色 f ，G 中 2-色圈 C 的类型与 G 的平面嵌入形式有关。如图 2(a)中所示的图，当其平面嵌入转化为图 2(c)时，2-色圈 C_1 是 2-色弦路圈，其中 $v_1x_1x_2x_3x_4x_5v_5y_5$ 为弦路，而 C,C_2 均是 2-色基本圈。若无特别声明，本文后面所言的 2-色圈皆指 2-色基本圈。对于 $f\in C_4^0(G)$ ，用 $C^2(f)$ 表示 f **中所有 2-色圈构成的集合**。若 $\exists f\in C_4^0(G)$ ，使得 $\left|f(C)\right|=2$ ，则称 C 是 G 的一个**可 2-色圈**。文中用 $C^2(G)$ 表示 G 的**全体可 2-色圈构成的集合**。显然，

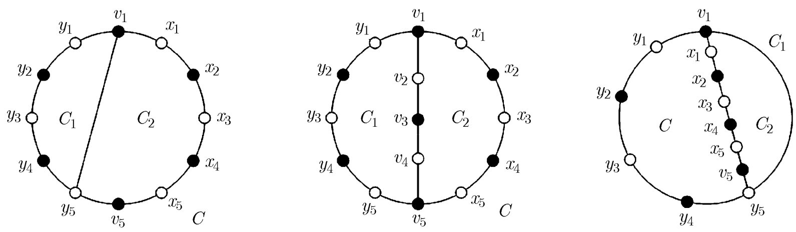
$$C^2(G)=\bigcup_{f\in C_4^0(G)}C^2(f) \tag{2}$$

设 G 是一个 $\delta(G)\geq 4$ 的 4-色极大平面图，f 是 G 的一个 4-着色。C_1,C_2 是 f 中的两个 2-色圈，如果 C_1 与 C_2 满足：

(1) $\left|f(C_1)\cap f(C_2)\right|=1$ ；

(2) $V_{C_1}^{\text{in}}\cap V(C_2)\neq\phi$ 且 $V_{C_1}^{\text{out}}\cap V(C_2)\neq\phi$ ，则称它们是**相交的**；否则，称 C_1,C_2 **不相交**。

设 $C^2(G)=\{C_1,C_2,\cdots,C_m\}$ ，$m\geq 2$ 。对其两个可 2-色圈 $C_1,C_t\in C^2(G)$ ，若存在可 2-色圈序列



(a) 2-色弦圈　(b) 2-色弦路圈　(c) 2-色基本圈

图 2 弦圈、弦路圈与基本圈

$C_1,C_2,\cdots,C_t$ 及对应的 4-着色 $f_1,f_2,\cdots,f_t$，使得 C_i 与 C_{i+1} $(1\le i\le t-1)$是 f_i 的相交 2-色圈，则称 C_1 与 C_t 是**相关的**。否则，称 C_1 与 C_t **不相关**。

图 3 所示的图中，C_1,C_2,C_3,C_4 是 $C^2(G)$ 中的 4 个可 2-色圈，f_1,f_2 是 G 的两个着色，C_1,C_2,C_3 是 f_1 的 2-色圈，C_3,C_4 是 f_2 的 2-色圈。由此推出：C_1 与 C_4 是相关的。

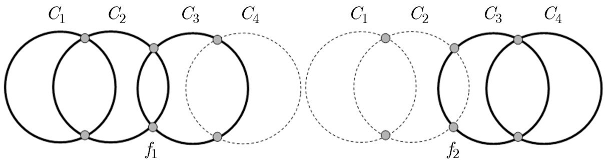


图 3 两个 2-色圈 C_1,C_4 相关的情况

2.2 2-色耳的相关定义与性质

设 G 是一个 4-色极大平面图，$f\in C_4^0(G)$，$C\in C^2(f)$，令 $f(C)=\{1,2\}$，x,y 是 C 上的一对同色顶点，则把从顶点 x 到顶点 y 之间的一条顶点数 ≥ 3，且非颜色 1 与颜色 2 构成的 2-色路 $P(x,y)$ 称为圈 C 的一个 **2-色耳**，或简称为圈 C 的一个**耳朵**，并把 x 与 y 均称为**耳根**。用 $Ed(C)$ 表示圈 C 上全体耳朵构成的集合。$Ed(C)$ 中的耳朵可分为两类：一类是圈 C 内的耳朵，称为**内耳**；另一类是圈 C 外的耳朵，称为**外耳**。

不失一般性，设圈 $f(C)=\{1,2\}$。把颜色 1 与颜色 3 构成的耳朵称为 **13-耳**，类似地有 **14-耳**，**23-耳**和 **24-耳**。我们把耳朵中非耳根的颜色称为**耳边色**。当然，耳边色要么为颜色 3，要么为颜色 4；

设 C 是 4-着色 f 的一个 2-色圈，$P(x,y)$ 与 $P(x',y')$ 是圈 C 的两只耳朵。若顶点 x,x' 是同色的，则称 $P(x,y)$ 与 $P(x',y')$ 为**同源耳**；若 $x=x'$，$y=y'$，则称 $P(x,y)$ 与 $P(x',y')$ 为**同根耳**。显然，同根耳属于同源耳，但同源耳不一定是同根耳。

设 $P(x,y)$ 与 $P(x',y')$ 是 2-色圈 C 的两只耳朵，$P(x,y)$ 与 $P(x',y')$ 称为**同色耳**，如果它们满足下列条件之一：

(1) $P(x,y)$ 与 $P(x',y')$ 均为内耳，或者外耳，且满足 $P(x,y)$ 所着的两种颜色与 $P(x',y')$ 的两种颜色相同。

(2) $P(x,y)$ 与 $P(x',y')$ 中一个为内耳，另一个为外耳，且为耳边色不同的同源耳。

否则，$P(x,y)$ 与 $P(x',y')$ 称为**异色耳**。

多个耳朵 $P_1,P_2,\cdots,P_m$ 称为**成路的**，如果 $P_1\cup P_2\cup\cdots\cup P_m$ ($m\ge 2$)是 G 的一条路。多个耳朵 $P_1,P_2,\cdots,P_m$ 称为**成圈的**，如果 $P_1\cup P_2\cup\cdots\cup P_m$ 是一个偶圈。把由 $Ed(C)$ 中耳朵形成所有可能的圈构成的集合记作 $Q(C)$。把只由内耳构成的圈称为**内耳圈**，C 的所有内耳圈构成的集合记为 $Q^i(C)$；把只由外耳构成的圈称为**外耳圈**，C 的所有外耳圈构成的集合记为 $Q^e(C)$；把既含内耳，又含外耳构成圈称为**混合耳圈**，C 的所有混合耳圈构成的集合记为 $Q^m(C)$。于是有

$$Q(C)=Q^i(C)\cup Q^e(C)\cup Q^m(C) \tag{3}$$

图 4 所示的图中，2-色圈 C 共有 5 只耳朵：P_1,P_2,P_3,P_4 和 P_5，其中 P_1,P_2,P_3,P_4 是同源耳，P_1 与 P_2 是同根耳；P_1,P_3,P_5 为内耳，P_2 与 P_4 为外耳；P_1,P_3,P_4 与 P_2,P_3,P_4 均成路；P_1,P_2 是成圈的。

定理 1 设 G 是一个最小度 ≥ 4 的 4-色极大平面图，$f\in C_4^0(G)$，且 C 是 f 的唯一 2-色圈，$f(C)=\{1,2\}$。则

(1) $Q^i(C)$ 与 $Q^e(C)$ 中的每个圈必含有异色耳；

(2)对 C 内所有 34-分支实施颜色互换后所得新着色 f^c 中含异于 C 的 2-色圈 C' 的充分必要条件是

$$C'\in Q^m(C) \tag{4}$$

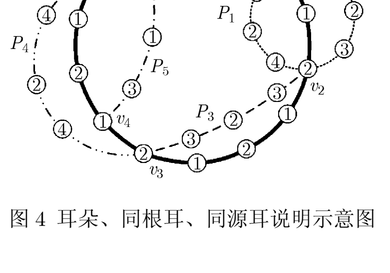


图 4 耳朵、同根耳、同源耳说明示意图

且构成C'的所有耳朵均为同色耳。

证明　(1)若在$Q^i(C)$或$Q^e(C)$中存在一个圈，设为C'，由同色耳构成，则由同色耳的定义知它们不仅是同源耳，而且任意两个耳朵颜色是相同的，故C'是f的一个 2-色圈，这与C是f的唯一 2-色圈矛盾，从而(1)获证；

(2)必要性。由C是f的唯一 2-色圈可知$C' \in Q^m(C)$。又注意到构成C'的所有耳朵均为同源耳。假设构成C'中的耳朵中有一对异色耳P, P'，则要么P, P'均在C内，要么P, P'均在C外，要么其中一个在C内，一个在C外。无论是哪种情形，均可推出C'不是f^c的 2-色圈，矛盾。

充分性。由$C' \in Q^m(C)$知C'是一个圈，且构成C'的所有耳朵是同源耳。又因为构成C'的所有耳朵均为同色耳，故当两只耳朵P, P'均在圈内或P, P'均在圈外时，它们在f下的着色恰有 2 种颜色；当P, P'一个属于圈内，一个属于圈外时，若其中一个由颜色 1(或 2)与颜色 3 的顶点构成，则另一个必由颜色 1(或 2)与颜色 4 的顶点构成，故在f^c下，P, P'只由两种颜色构成，即证明了C'是f^c的 2-色圈。

证毕

设C是 4-色极大平面图G中关于着色$f \in C_4^0(G)$的唯一 2-色圈，$f(C)=\{1,2\}$，对C内所有 34-分支实施颜色互换后所得新着色记为f^c。若f^c中含有至少两个 2-色圈，则除了C外，其余的 2-色圈C'是由C的同色耳构成，可具有如下情况：

(1)若C'只有两个同色耳朵，则这两个耳朵分别是圈内耳和圈外耳，结构如图 5(a)所示；

(2)若C'有 3 个同色耳朵，其可能的结构如图 5(b), 5(c)所示；

(3)若C'有 4 个同色耳朵，其可能的结构如图 5(d), 5(e)所示；

(4)对C'含有更多个同色耳朵，其一般可能的结构如图 5(f)所示。

3　σ-运算与σ-特征图

文献[26]中引入了树着色与圈着色的概念，为方便，在此重述如下。设G是一个 4-色极大平面图，$f \in C_4^0(G)$，$\{1,2,3,4\}$为颜色集。若$\exists i,t \in \{1,2,3,4\}$，使得$G_{it}^f$中含圈，则称$f$为**圈着色**，并称$G$为**可圈着色的**；反之，若$\forall i,t \in \{1,2,3,4\}$，$G_{it}^f$中均无圈，则称$f$为图$G$的**树着色**，并称$G$为**可树着色的**；若$G$是可树着色，但非可圈着色，则称$G$为**纯树着色的**；若$G$是可圈着色，但非可树着色，则称$G$为**纯圈着色的**；若$G$既是可圈着色，又是可树着色，则称$G$为**混合型着色的**。由圈着色与树着色的定义，对任一 4-色极大平面图G，可将$C_4^0(G)$中的着色分为两类：圈着色与树着色。

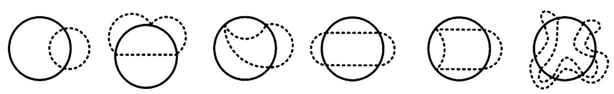


图 5 成圈的同色耳结构示意图，其中实线表示C

3.1　σ-运算与 Kempe 等价类

设f是极大平面图G的一个 4-着色，$\{1,2,3,4\}$为颜色集，C是f的一个 2-色圈，且$f(C)=\{1,2\}$，f关于C的σ**-运算**，记作$\sigma(f,C)$，在已知圈C时，也可简记为$\sigma(f)$，是指：将C内所有颜色 3 与颜色 4 的顶点颜色互换，同时保持其它顶点着色不变的一种导色运算。显然，基于f的σ-运算是将f变换成G的另一个圈着色，记作f^c，即

$$\sigma(f,C)=f^c \tag{5}$$

并称f^c与f为**基于C的互补着色**。在不考虑圈C时，式(5)可用$\sigma(f)=f^c$来表示。显然，若C内仅有一个 34-分支，则σ-运算即为 Kempe 变换；若C内所含 34-分支数$m \ge 2$，则一次σ-运算包含了m次 Kempe 变换，故，f^c与f是 Kempe 等价的。

对$C_4^0(G)$中的任一着色f_0，称

$$F^{f_0}(G) \triangleq \left\{f; f \text{ 与 } f_0 \text{ 是 Kempe 等价的}, f \in C_4^0(G)\right\} \tag{6}$$

为f_0的 **Kempe 等价类**。

图 6 中所示的 11-阶极大平面图G共有 8 种 4-着色$f_1 \sim f_8$。对前 4 种f_1, f_2, f_3, f_4，易验证$\sigma(f_1,C_1)=f_2$；$\sigma(f_2,C_2)=f_3$；$\sigma(f_3,C_3)=f_4$。另外，G共有 7 个可 2-色圈$C_1 \sim C_7$，即$\left|C^2(G)\right|=7$。

由σ-运算定义可直接推出定理 2：

定理 2　设f是G的一个 4-着色，C是f的一个 2-色圈，则

$$\sigma(\sigma(f,C),C)=f \tag{7}$$

σ-运算的目的是：由G的一个 4-着色f出发，通过不断地实施σ-运算，导出$C_4^0(G)$中f的 Kempe 等价类$F^f(G)$。业已得知，对于极大平面图G，从$C_5^0(G)$中的任一 5-着色出发，可导出$C_5^0(G)$中全部 5-着色[19]，而对$C_4^0(G)$，情况并非如此，可能含多个 Kempe 等价类，将在第 4 节中给予详细讨论。

3.2　σ-特征图的定义

针对σ-运算，引入σ-特征图，它使得$C_4^0(G)$中着色之间的关系清楚直观。

设G是一个 4-色极大平面图，$C_4^0(G)=\{f_1, f_2, \cdots, f_n\}$。$G$的$\sigma$**-特征图**，记作$G_4^\sigma$，顶点集$V(G_4^\sigma)=\{f_1, f_2, \cdots, f_n\}$，$V(G_4^\sigma)$中两个顶点$f_i$与$f_t$相邻当且

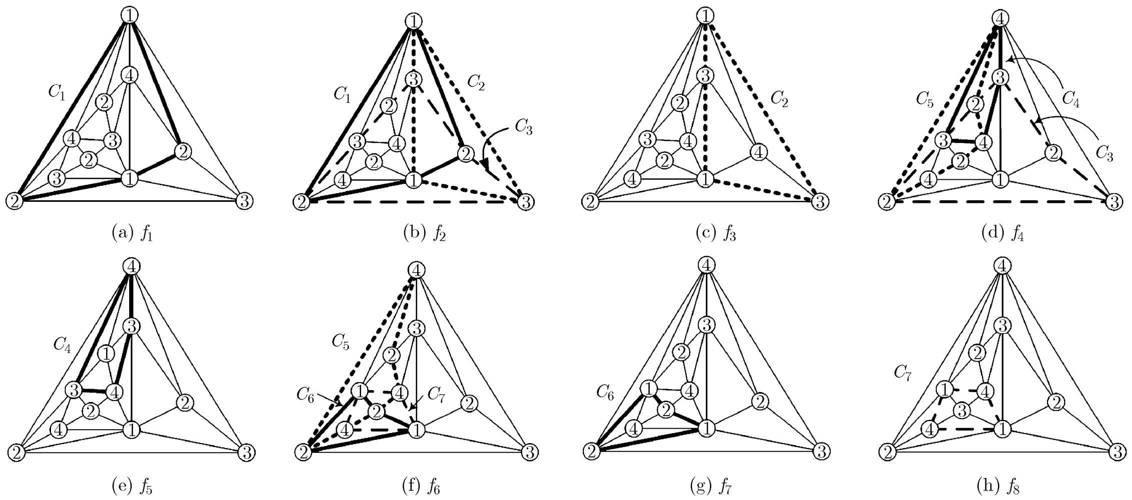


图 6 σ -运算示例

仅当它们在 G 中关于某 2-色圈 C 为互补着色，$i,t=1,2,\cdots,n, i\neq t$ 。由此定义，可视 G_4^{σ} 为一个**边标定图**，其边上的标号即为导致两个着色互补的圈 C 。在只考虑拓扑结构的情况下，可略去边上的标号。

图 6 所示的图 G 共有 8 种着色，其 σ -特征图，及相应的拓扑结构如图 7(a), 7(b)所示，显然 G_4^{σ} 是一个连通图，且是一棵树，故只需知道 G 的任一 4-着色，即可通过 σ -运算得到 $C_4^0(G)$ 其它的 4-着色。正二十面体的全部 10 种着色均为树着色，故其 σ -特征图是由 10 个孤立顶点构成的 10-阶空图。

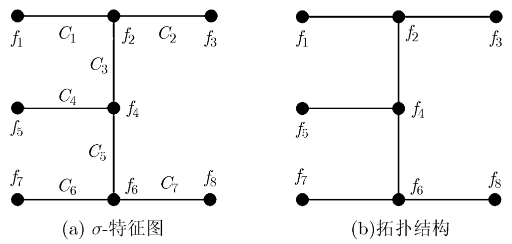


图 7 图 6 中所示图 G 的 σ -特征图

3.3 σ -特征图的基本性质

设 G 是 4-色极大平面图，如果 $\left|C_4^0(G)\right|=1$，则称 G 是**唯一 4-色极大平面图**。

定理 3 设 G 是 4-色极大平面图，G_4^{σ} 是它的 σ -特征图。则有：

(1) G 是唯一 4-色极大平面图当且仅当 $G_4^{\sigma}\cong K_1$；

(2) G_4^{σ} 中任一顶点 f，$d_{G_4^{\sigma}}(f)=k$ 的充分必要条件是 f 中含 k 个 2-色圈；

(3)若 $\exists f\in C_4^0(G)$，它含有 $k\ (\geq 1)$个 2-色圈，即 $d_{G_4^{\sigma}}(f)=k$，则有

$$\left|C_4^0(G)\right|\geq k+1 \tag{8}$$

对于阶数不超过 11 的所有极大平面图，树着色的数目约占 2%[26]。故我们推测对最小度≥ 4 的极大平面图集，树着色数相比圈着色数来说非常少，因此，给定 4-色极大平面图的一个圈着色，很有可能通过 σ -运算将 $C_4^0(G)$ 中的其它着色推导出来。

定理 4 设 G 是一个 4-色极大平面图，若 G_4^{σ} 连通，则求出 G 的全部 4-着色的算法只比求出 G 中一种 4-着色多$\left|V(G_4^{\sigma})\right|-1$ 次。

根据定理 4 可知，若 G 是 4-色 Kempe 极大平面图，求 G 的全部 4-着色算法复杂度与求出 G 中一个 4-着色的算法复杂度是等价的。

G_4^{σ} 与 $C_4^0(G)$ 息息相关，研究 G_4^{σ} 的结构是揭示 $C_4^0(G)$ 本质的一项基础性工作。本文对 G_4^{σ} 结构的研究，将通过每个着色 f 的 Kempe 等价类 $F^f(G)$ 来展开。

定理 5 设 G 是一个 $\delta(G)\geq 4$ 的 4-色极大平面图，$f\in C_4^0(G)$ 。则 G_4^{σ} 不含三角形。

证明 假设 G_4^{σ} 中含三角形 $f_1f_2f_3f_1$，不失一般性令

$$\sigma(f_1,C_1)=f_2,\ \sigma(f_1,C_2)=f_3,\ \sigma(f_2,C_3)=f_3 \tag{9}$$

故 C_1, C_2 是 f_1 的2-色圈，C_1, C_3 是 f_2 的2-色圈，C_2, C_3 是 f_3 的 2-色圈，如图 8(a)所示。设颜色集 $C(4)=\{1,2,3,4\}$ 。用 $V_{C_i}^{*\mathrm{in}}$ 表示 C_i 内部中所有与 C_i 上颜色不同的顶点构成的集合，$i=1,2$ 。

如下 2 种情况给予证明：

情况 1 在 f_1 下，C_1, C_2 不相交，如图 8(b)所示。其中，C_1, C_2 可以着色相同，也可以不同，其证明过程相同。

显然，$\forall u\in V_{C_1}^{*\mathrm{in}}$，有 $f_1(u)\neq f_2(u)$；$\forall v\in V(G)\setminus V_{C_1}^{*\mathrm{in}}$，有 $f_1(v)=f_2(v)$，如图 8(b)–图 8(c)所示。同

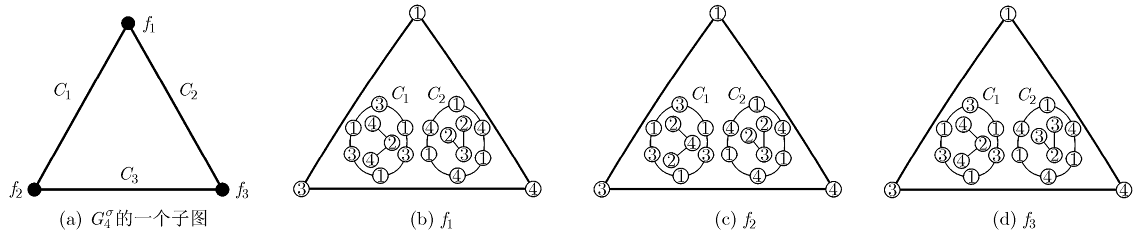


图 8　C_1, C_2 不相交的情况

理，对 $\forall u \in V_{C_2}^{*\text{in}}$，有 $f_1(u) \neq f_3(u)$；但 $\forall v \in V(G) \setminus V_{C_2}^{*\text{in}}$，有 $f_1(v) = f_3(v)$，如图 8(b)和图 8(d)所示。故有，$\forall u \in V_{C_1}^{*\text{in}} \cup V_{C_2}^{*\text{in}}$，有 $f_2(u) \neq f_3(u)$，$\forall v \in V(G) \setminus V_{C_1}^{*\text{in}} \cup V_{C_2}^{*\text{in}}$，有 $f_2(v) = f_3(v)$。显然，f_2, f_3 中不存在能够满足 $V_{C_3}^{*\text{in}} = V_{C_1}^{*\text{in}} \cup V_{C_2}^{*\text{in}}$ 的 2-色圈 C_3，矛盾。

情况 2　在 f_1 下，若 C_1, C_2 相交，则 $f_1(C_1) \neq f_1(C_2)$，不失一般性，设 $f_1(C_1) = \{1,2\}$，$f_1(C_2) = \{1,3\}$，如图 9(a)所示。取 $u \in V_{C_1}^{*\text{in}} \cap V(C_2)$，$v \in V_{C_2}^{*\text{in}} \cap V(C_1)$，则 $f_1(u) = 3$，$f_1(v) = 2$；$f_2(u) = 4$，$f_2(v) = 2$；$f_3(u) = 3$，$f_3(v) = 4$。显然，f_2 中不存在 2-色圈 C_3，使得 f_2 和 f_3 为基于 C_3 的互补着色，矛盾。

基于上述两种情况，本定理获证。

由于任意最小度 ≥ 4 的极大平面图 G 的 σ-特征图 G_4^σ 中不含三角形，这就意味着，若将 G_4^σ 中每个顶点 f 的闭邻域，记作 $\overline{N^f(G)}$，则其导出子图 $G_4^\sigma\left[\overline{N^f(G)}\right]$ 是一个星图，即有：

推论 1　设 G 是一个 $\delta(G) \geq 4$ 的 4-色极大平面图。则对任一圈着色 $f \in C_4^0(G)$，$G_4^\sigma\left[\overline{N^f(G)}\right]$ 是一个星图。

定理 6　设 G 是一个 4-色极大平面图，f 是 G 的一个 4-着色。C 是 f 的一个 2-色圈，且 f 是 G_4^σ 中的 1 度顶点，若 $\sigma(f) = f^c$ 在 G_4^σ 中的度数 ≥ 2，则一定存在着色 $f' \in C_4^0(G)$，使得 C 不是 f' 的 2-色圈。

证明　因 $\left|C^2(f)\right| = 1$，令 C 是 f 的 2-色圈，且设 $f(C) = \{1,2\}$，如图 10(a)所示。由于 f^c 中至少含两个 2-色圈，故在 C 上至少存在一对基于着色 f 的同色点对，记作 v_1, v_2，令 $f(v_1) = f(v_2) = 1$，使得基于 v_1, v_2 的圈外存在一个 14-耳，圈内存在一个 13-耳，如图 10(a)所示。现实施关于 C 的 σ-运算，得到着色 f^c。f^c 中含至少两个 2-色圈 C 与 C'，如图 10(b)所示。于是，对 C' 内的 23-分支实施 σ-运算，得到一个新的着色 f', $\left|f'(C)\right| = 3$，如图 10(c)所示。本定理获证。

定理 7　设 G 是一个 $\delta(G) \geq 4$ 的 4-色极大平面图，则 G_4^σ 中任意一对相邻顶点的度数不可能是 1 和 2。

证明　假设 G_4^σ 中存在度数分别为 1 和 2 的两个相邻顶点 f_1, f_2，则可令 f_1 只含一个 2-色圈 C_1，f_2 恰含两个相交的 2-色圈 C_1 和 C_2，且 f_1 与 f_2 关于 C_1 是互补的。不失一般性，设 $f_1(C_1) = \{1,2\}$，$f_2(C_1) = \{1,2\}$，$f_2(C_2) = \{1,3\}$。将 C_1 及其内部顶点构成的集合与 C_1 及其外部顶点构成的集合在 G 中的导出子图分别记为 $G_1^{C_1}, G_2^{C_1}$。用 r_{ij} 表示 $G_1^{C_1}$ 中 ij-分支数，$i,j \in \{1,2,3,4\}, i \neq j$。

在 f_1 下，G 仅含一个 2-色圈 C_1，则 G_{34} 含 2 个分支，G_{13}, G_{14}, G_{23} 和 G_{24} 分别是连通的。此时，$G_1^{C_1}$ 中的 r_{14} 个 14-分支被 $G_2^{C_1}$ 中的 $r_{14}-1$ 条 14-路连接成一个分支，$G_1^{C_1}$ 中的 r_{13} 个 13-分支被 $G_2^{C_1}$ 中的 $r_{13}-1$ 条 13-路连接成一个分支。

将 $G_1^{C_1}$ 中 34-分支实施颜色互换，所得着色为 f_2。在 f_2 下，$G_1^{C_1}$ 中 13-分支数变为 r_{14} 个，$G_2^{C_1}$ 中仍含 $r_{13}-1$ 条两端点均在 C_1 上的 13-路，由于这些 13-路将 r_{14} 个 $G_1^{C_1}$ 中的 13-分支连接成一个连通分支且含一个 13-圈 C_2，因此，$r_{13} > r_{14}$；$G_1^{C_1}$ 中 14-分支数变为 r_{13} 个，$G_2^{C_1}$ 中仍含 $r_{14}-1$ 条两端点均在 C_1 上的 14-路，由于 $r_{13} > r_{14}$，故这些 $r_{14}-1$ 条 14-路不能将

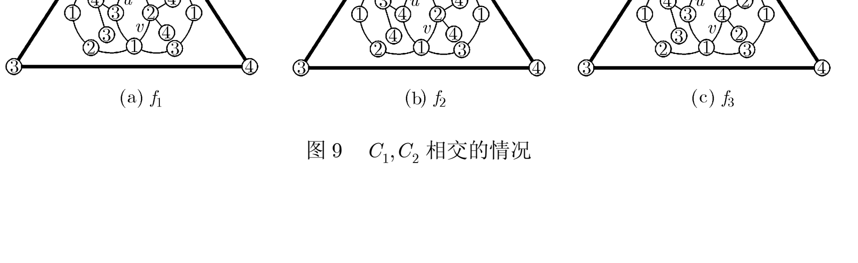
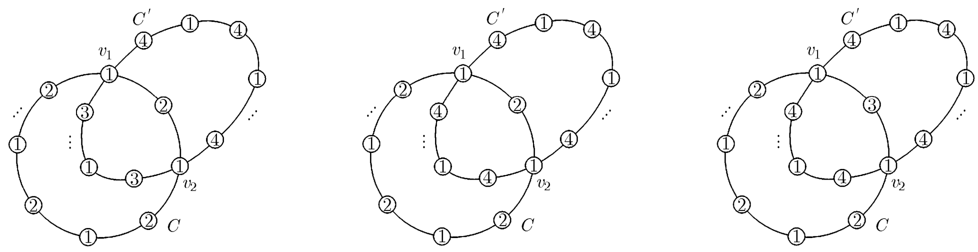


图 9　C_1, C_2 相交的情况



(a) f　(b) f^c　(c) f'

图 10 定理 6 证明示意图

$G_1^{C_1}$ 中的 r_{13} 个 14-分支连接成一个连通分支，因此基于 f_2 的 G_{14} 不连通，即 G_{23} 中含 2-色圈，矛盾。证毕

G_4^σ 不连通情况是研究重点，将在第 4 节研究。

4 非 Kempe 图的 Kempe 等价类类型

设 G 是一个 4-色极大平面图，$f, f' \in C_4^0(G)$，如果 f 与 f' 为非 Kempe 等价的，则说明从 $C^2(f)$ 中任意 2-色圈出发，通过连续 σ-运算，不能获得 f'。出现这种情况的原因与 $C^2(f)$ 有关，可归结为 3 种情况：(1) $C^2(f)=\phi$，即 f 是树着色；(2) $C^2(f)$ 中含 **2-色不变圈**(其定义稍后给出)；(3) $C^2(f)$ 中含**循环 2-色圈**(其定义稍后给出)。针对这 3 种情况，相应地将非 Kempe 图的 Kempe 等价类分为 3 类：树型、圈型和循环圈型。下面，我们对这 3 种类型逐一介绍。为方便，文中用 $\omega(G_4^\sigma)$ 表示 $C_4^0(G)$ 中 Kempe 等价类的数目。

4.1 树型 Kempe 等价类

若 f 是 G 的一个树着色，则 $C^2(f)=\phi$，即 f 中全部 6 个 2-色导出子图是连通的，因此，$F^f(G)=\{f\}$。我们把非 Kempe 图 G 的这种 Kempe 等价类称为**树型 Kempe 等价类**，并把 G 称为**树型极大平面图**。因此，$\forall f'(\neq f) \in C_4^0(G)$，有

$$f' \notin F^f(G) \tag{10}$$

由此推出定理 8：

定理 8 设 G 是一个非唯一 4-色极大平面图。则 $\omega(G_4^\sigma) \geq 2$；若 G 是纯树着色的，则 $\omega(G_4^\sigma) = \left|C_4^0(G)\right|$；若 G 是混合型着色，且含树着色的数目为 t 个，则 $\omega(G_4^\sigma) \geq t+1$。

4.2 圈型 Kempe 等价类

设 G 是一个 4-色极大平面图，$f \in C_4^0(G)$，$C \in C^2(f)$。若 $\forall f' \in F^f(G)$，均有 $\left|f'(C)\right|=2$，则称 C 为 f 的一个 **2-色不变圈**，并称 f 是 G 的一个 **2-色不变圈着色**，G 为基于 C 的**圈型极大平面图**。由此定义知，在 f 下，若实施关于 C 的 σ-运算，所得之 4-着色记为 f^c。则

$$C^2(f) = C^2(f^c) \tag{11}$$

式(11)中，不考虑 $C^2(f)$ 与 $C^2(f^c)$ 中 2-色圈的颜色，即某 2-色圈 C 可能在 f 与 f^c 下的颜色不同。

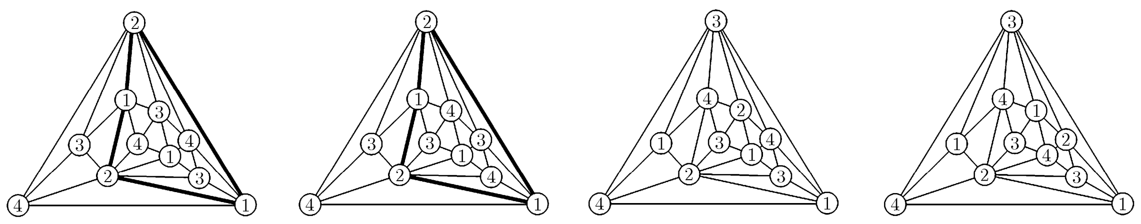
当然，满足式(11)的 f^c 也是图 G 的一个 2-色不变圈着色。图 11 中所示的图是一个圈型极大平面图，共有 4 个着色，其中 f_1 与 f_2 为一对互补的 2-色不变圈着色，它们所含唯一的 2-色不变圈为 12-圈。

图 12 中分别给出了两个都含 2 个 2-色不变圈的极大平面图，其中第 1 个图中的 2 个 2-色不变圈有两个公共顶点。图 12(a)–图 12(j)给出了该图的所有 4-着色，且在图 12(k)给出了它的 σ-特征图。图 12(l)给出了含不相交的 2 个 2-色不变圈(用粗线标记)的极大平面图。

设 G 是一个 4-色极大平面图，$f \in C_4^0(G)$，$C \in C^2(f)$。若 $\exists f' \in C_4^0(G)$，使

$$\left|f'(C)\right| \geq 3 \tag{12}$$

则称圈 C 是**可破的**。



(a) f_1，圈着色，含12-圈　(b) f_2，圈着色，含12-圈　(c) f_3，树着色　(d) f_4，树着色

图 11 只含有一个 2-色不变圈的圈型极大平面图

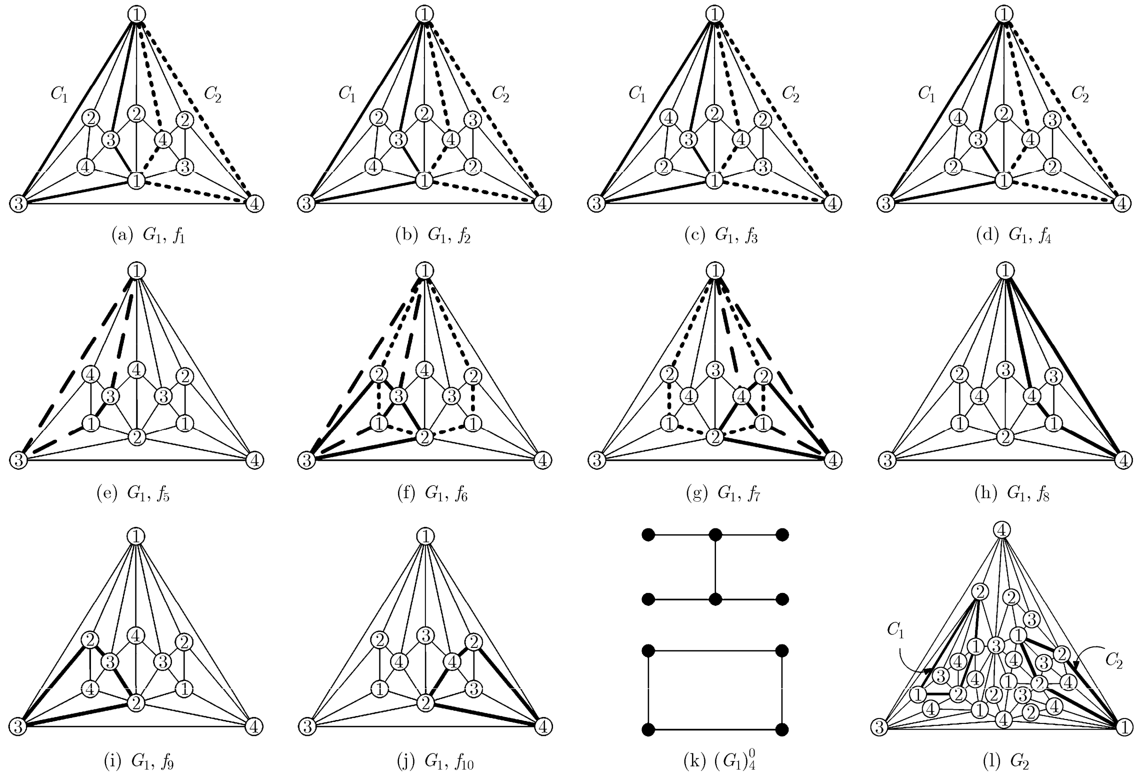


图 12 含 2 个 2-色不变圈的圈型极大平面图

定理 9 设G是一个$\delta(G)\ge 4$的 4-色极大平面图，C_1与C_t是G的两个可 2-色圈，且是相关的，则C_1与C_t均为可破的。

证明 由可 2-色圈相关的定义，基于σ-运算，易证本定理成立。

对满足定理 9 的两个 2-色圈，显然由σ-运算知这两个圈均为可破的。那么，对于一个含 2-色不变圈C的f，基于σ-运算是不能破圈C的，即利用σ-运算，无法在$C_4^0(G)$中找到f'，使f'满足式(12)。这就是说，f与f'为**非 Kempe 等价的**。导致这种不等价的根本原因是 2-色不变圈所致，故把$F^f(G)$称为**圈型 Kempe 等价类**。对于恰含$k\,(\ge 2)$个 2-色不变圈的 4-着色集，它们在G_4^σ中的顶点导出子图是一个k-维超立方体。所谓**t-维超立方体图**，简称**t-维超立方体**，记作B^t，其顶点集为

$$V\left(B^t\right)=\left\{\left(x_1,x_2,\cdots,x_t\right);x_i\in B=\{0,1\}\right\}\tag{13}$$

B^t中的两个顶点X_1与X_2相邻当且仅当

$$d_H\left(X_1,X_2\right)=1\tag{14}$$

其中$d_H\left(X_1,X_2\right)$表示X_1与X_2的 Hamming 距离，即两个向量对应分量不同元素对的数目。由此定义易知t-维超立方体图B^t是一个2^t阶的t-正则图，因此有

$$\left|E\left(B^t\right)\right|=t\cdot 2^{t-1}\tag{15}$$

定理 10 设G是一个$\delta(G)\ge 4$的 4-色极大平面图，f是G的一个恰含k个 2-色圈，且均为 2-色不变圈的 4-着色，则$\left|F^f(G)\right|=2^k$，且$G_4^\sigma\left[F^f(G)\right]=B^k$。

证明 设f是G的一个恰含k个 2-色圈，且均为 2-色不变圈的 4-着色，其中$C_1,C_2,\cdots,C_k$是它的k个 2-色不变圈。对每个$C_i\,(1\le i\le k)$，若实施σ-运算，则用 1 表示，若没有实施σ-运算，则用 0 表示。于是由 2-色不变圈构成的序列$C_1,C_2,\cdots,C_k$按照是否实施关于某个圈的σ-运算与长度为k的 0-1 序列集$\{(x_1,x_2,\cdots,x_k);\ \ x_i=0,1,i=1,2,\cdots,k\}$建立 1-1 对应关系：对$C_i\,(1\le i\le k)$施行$\sigma$-运算当且仅当$x_i=1$。因为每次对$C_i\,(1\le i\le k)$施行一次$\sigma$-运算恰好对应$G$中的一个着色，也就是对应$\{(x_1,x_2,\cdots,x_k);x_i=0,1,i=1,2,\cdots,k\}$中的一个 0-1 序列。因此，$f$所在$G_4^\sigma$的连通分支上至少有$2^k$个着色；

另一方面，不失一般性，设f所对应的长度为k的 0-1 序列为$(0,0,\cdots,0)$，则对每个$C_i\,(1\le i\le k)$施行σ-运算后所得到的着色记为$f_i'\,(1\le i\le k)$，即f恰好导出k个互补着色。同理可证，每个$f_i'\,(1\le i\le k)$恰好导出k个互补着色。进而可证2^k个着色中的每个 4-着色恰好导出k个互补着色。注意到每个 4-着色可导出它的互补着色当且仅当它们对应的长度为k的 0-1 序列之间的Haming距离等于 1，从而

证明了这2^k个4-着色对应的2^k个0-1序列构成的图是超立方体图。

显然，这2^k个4-着色是封闭的，即不能通过σ-运算导出这2^k个之外的任何一个4-着色。从而本定理获证。

一个自然的问题是：**任意4-着色f中的任意2-色圈是否可破?**答案是肯定的。基于定理9，我们实际上只需再证明2-色不变圈的可破性，将在本系列后续文章证明。在此，先作为一个猜想给出：

猜想 2 若G是一个$\delta(G)\ge 4$的4-色极大平面图，$f\in C_4^0(G)$，$C\in C^2(f)$，则C可破。

4.3 循环圈型 Kempe 等价类

设G是一个$\delta(G)\ge 4$的4-色非Kempe极大平面图，$\mathbb{C}\subseteq C^2(G)$。如果(1)$\forall C_1,C_2\in\mathbb{C}$，$C_1$与$C_2$相关；(2)$|\mathbb{C}|\ge 2$；(3)$\mathbb{C}$是极大相关圈集：即在$C^2(G)\setminus\mathbb{C}$中，不存在任何可2-色圈$C'$，它与$\mathbb{C}$中的任一可2-色圈相关。则把$\mathbb{C}$中的可2-色圈均称为**循环2-色圈**，$F^f(G)$中的着色均称为**循环圈着色**，其中$f\in C_4^0(G)$含$C\in\mathbb{C}$，且$f$使得$C$与$\mathbb{C}\setminus C$中的某个2-色圈相关；$\mathbb{C}$称为$F^f(G)$的**循环2-色圈集**；用$F_{\mathbb{C}}^f(G)$表示从$f$出发，实施关于$\mathbb{C}$中2-色圈的$\sigma$-运算得到的所有着色构成之集，并称之为关于$\mathbb{C}$的**循环2-色圈集**；且若$F^f(G)$不含2-色不变圈着色，则称$F^f(G)$所在的Kempe等价类为**循环圈型Kempe等价类**。若G含循环圈型Kempe等价类，则称G为**循环圈型极大平面图**。

图13所示两个图G与H，相应4-着色分别为f，g。容易验证，f既是2-色不变圈着色，又是循环圈着色，具体讨论如下：

(1) f是基于$C_1=v_1v_2v_3v_4v_1$的2-色不变圈着色；

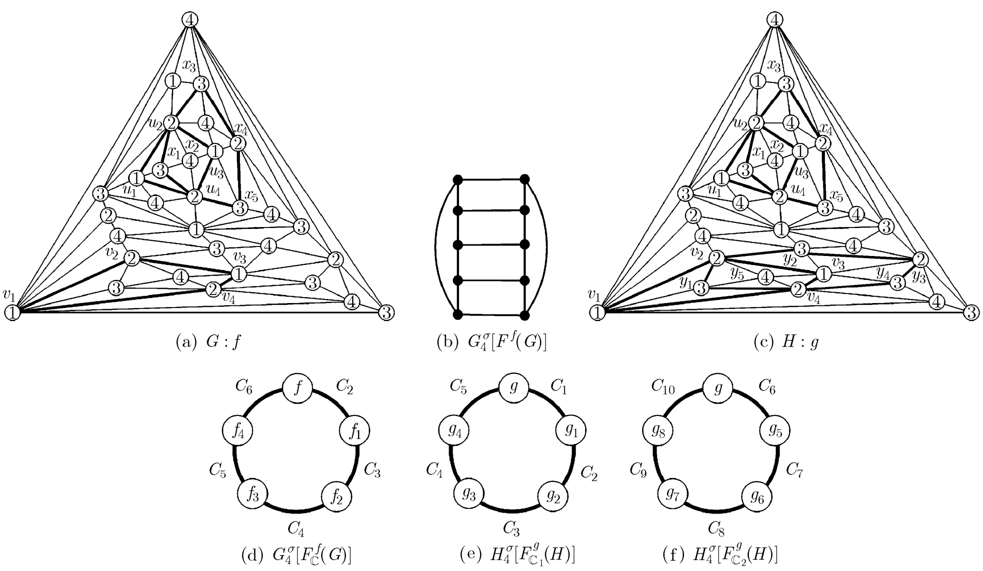
(2) f是基于循环2-色圈集$\mathbb{C}=\{C_2,C_3,C_4,C_5,C_6\}$的循环圈着色，其中$C_2=u_1u_2u_3u_4$，$C_3=u_2x_3x_4x_5u_4x_2$，$C_4=x_1u_2u_3u_4$，$C_5=u_1u_2x_2u_4$，$C_6=x_1u_2x_3x_4x_5u_4$；关于$\mathbb{C}$的循环圈着色集$F_{\mathbb{C}}^f(G)=\{f,f_1,f_2,f_3,f_4\}$，其中，$f_1=\sigma(f,C_2)$，$f_2=\sigma(f_1,C_3)$，$f_3=\sigma(f_2,C_4)$，$f_4=\sigma(f_3,C_5)$，$f=\sigma(f_4,C_6)$；$F_{\mathbb{C}}^f(G)$在$G_4^\sigma$的导出子图如图13(d)所示。

g是图H的一个4-着色，它含2个循环2-色圈集：$\mathbb{C}_1$与$\mathbb{C}_2$，其中$\mathbb{C}_1=\{C_1,C_2,C_3,C_4,C_5\}$，$\mathbb{C}_2=\{C_6,C_7,C_8,C_9,C_{10}\}$，这里，$C_1=v_1v_2v_3v_4$，$C_2=y_5v_2y_2y_3y_4v_4$，$C_3=y_1v_2v_3v_4$，$C_4=v_1v_2y_5v_4$，$C_5=y_1v_2y_2y_3y_4v_4$，$C_6=u_1u_2u_3u_4$，$C_7=x_2u_2x_3x_4x_5u_4$，$C_8=x_1u_2u_3u_4$，$C_9=u_1u_2x_2u_4$，$C_{10}=x_1u_2x_3x_4x_5u_4$。每个循环2-色圈集对应的循环圈着色集分别为：$F_{\mathbb{C}_1}^g(H)=\{g,g_1,g_2,g_3,g_4\}$与$F_{\mathbb{C}_2}^g(H)=\{g,g_5,g_6,g_7,g_8]$，$F_{\mathbb{C}_1}^g(H)$中，$g_1=\sigma(g,C_1)$，$g_2=\sigma(g_1,C_2)$，$g_3=\sigma(g_2,C_3)$，$g_4=\sigma(g_3,C_4)$，$g=\sigma(g_4,C_5)$，$F_{\mathbb{C}_2}^g(H)$中，$g_5=\sigma(g,C_6)$，$g_6=\sigma(g_5,C_7)$，$g_7=\sigma(g_6,C_8)$，$g_8=\sigma(g_7,C_9)$，$g=\sigma(g_8,C_{10})$；$F_{\mathbb{C}_1}^g(H)$在$H_4^\sigma$的导出子图如图13(e)所示，$F_{\mathbb{C}_2}^g(H)$在$H_4^\sigma$的导出子图如图13(f)所示。

(3) $F^f(G)$构成一个圈型Kempe等价类，f所在的G_4^σ的连通分支如图13(b)所示。$F^g(H)$构成一个循环圈型Kempe等价类。

由图13两个例子可以看出，对$C_4^0(G)$中的一个4-着色f，由它导出的Kempe等价类中，可能具有如下几种类型：

纯圈型，即含1个或多个2-色不变圈，如图11，图12所示的图；



(a) $G:f$　(b) $G_4^\sigma[F^f(G)]$　(c) $H:g$

(d) $G_4^\sigma[F_{\mathbb{C}}^f(G)]$　(e) $H_4^\sigma[F_{\mathbb{C}_1}^g(H)]$　(f) $H_4^\sigma[F_{\mathbb{C}_2}^g(H)]$

图13 圈型及循环型圈型Kempe等价类的两个例子

混合型，即既含 2-色不变圈型，又含循环圈型，如图 13(a)；

纯循环圈型，即只含 1 个或多个循环 2-色圈，如图 13(c)中所给出的 4-着色 g，它同时含 2 个循环圈型。

注 1　有些图在不同着色下含相同的某 2-色圈，但这些着色并不属于同一 Kempe 等价类。

注 2　由上述给出 3 种 Kempe 等价类可知，在一个给定的最小度 ≥ 4 的极大平面图 G 中，$C_4^0(G)$ 中可能存在 1~3 种 Kempe 等价类，且可能存在多个同种类型的 Kempe 等价类。如正 20 面体含有 10 个树型等价类。

σ-运算无法从 G 中的一个 Kempe 等价类导出另一个 Kempe 等价类。为了解决此问题，我们提出了解决此瓶颈的两种方法：破圈法与破树法。为此，需要对圈型极大平面图，以及循环圈型极大平面图展开深入研究，我们将在后续文章中给出。

5　Kempe 图

如果一个最小度 ≥ 4 的 4-色极大平面图 G 是 Kempe 图，则只需知道 G 中的一个 4-着色，便可通过 σ-运算，导出 $C_4^0(G)$ 中的其余 4-着色。这是我们最希望的一种情况。为了刻画此类图的特征，本节提出了基于扩多米诺构形运算的 Kempe 图递推构造方法，并提出了两个猜想。

5.1 Kempe 图猜想

对于一个最小度 ≥ 4 的 4-色极大平面图 G，由上节可知，非 Kempe 图的 Kempe 等价类有树型，圈型，循环圈型等 3 类。

猜想 3　设 G 一个最小度 ≥ 4 的 4-色极大平面图。则 G 是 Kempe 图当且仅当 G 不是树型，不是圈型，也不是循环圈型。

此猜想与唯一 4-色极大平面图猜想息息相关。若唯一 4-色极大平面图猜想成立，即每个唯一 4-色极大平面图是递归极大平面图[26]，则最小度 ≥ 4 的 4-色极大平面图 G 至少有 2 种不同的 4-着色。若 G 是树型，由于树着色不能通过 σ-运算到达 G 的其它 4-着色，因此 G 一定不是 Kempe 图。

此猜想也与猜想 2 相关。假设 $\exists f \in C_4^0(G)$，且 C 是满足 $|f(C)|=2$ 的 2-色不变圈。若猜想 2 成立，即 C 是可破的，于是 $\exists f' \in C_4^0(G)$，使 $|f'(C)| \geq 3$。但 f 与 f' 不能通过 σ-运算到达，故，G 不是 Kempe 图。

即使猜想 3 成立，也不能仅从 Kempe 图所含等价类的类型来确定 Kempe 图的特征。为深入研究 Kempe 图的特征，我们提出了**多米诺递推构造法**。

5.2 Kempe 图的构造

由本系列文章(2)[25]知：一个最小度 ≥ 4 的 $n(\geq 9)$-阶极大平面图 G，其祖先图集中必含 $(n-2)$-阶，或 $(n-3)$-阶最小度 ≥ 4 的极大平面图。换言之，G 中至少存在图 14 中的 5 个基本多米诺构形之一。为方便，本系列文章将 5 个基本多米诺构形分别标记为 $W_4^1, W_5^1, W_4^2, W_5^2, W_6^2$，如图 14 所示。

设 G 是一个最小度 ≥ 4 的 $n(\geq 7)$-阶非可分 4-色极大平面图，P_3 是 G 的一条 2-长路。在 G 中基于 P_3 实施扩 4-轮运算，必产生一个新的多米诺构形 W_4^1，故此运算也称为**扩 W_4^1-运算**，所得之图记作 $\zeta^{W_4^1}(G)$；类似地，把在 G 中通过多米诺扩轮运算得到图 14(b)–图 14(e)所示的基本多米诺构形的过程分别称为**扩 W_5^1-运算**，**扩 W_4^2-运算**，**扩 W_5^2-运算**和**扩 W_6^2-运算**，所得之图分别记作 $\zeta^{W_5^1}(G)$，$\zeta^{W_4^2}(G)$，$\zeta^{W_5^2}(G)$ 和 $\zeta^{W_6^2}(G)$。

考察基于 f 的通过多米诺扩轮运算所得自然着色 f'，可按照如下 3 种情况易证明下述定理 11：f 与 f' 中有一个为树型；f 与 f' 中有一个为圈型；f 与 f' 有一个为循环圈型：

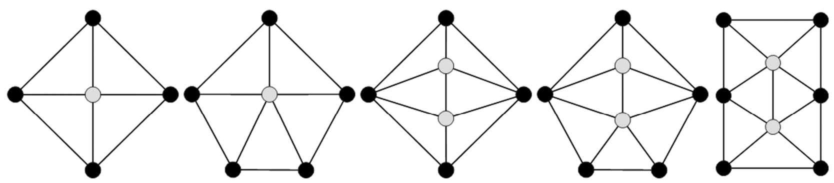
定理 11　设 G 是一个 $\delta(G) \geq 4$ 的 4-色极大平面图。则 $\zeta^{W_4^1}(G)$，$\zeta^{W_4^2}(G)$ 是 Kempe 图当且仅当 G 是 Kempe 图。

而对于 $\zeta^{W_5^1}(G)$ 与 $\zeta^{W_5^2}(G)$，我们在此提出如下猜想：

猜想 4　设 G 一个最小度 ≥ 4 的 4-色的 Kempe 极大平面图。则 $\zeta^{W_5^1}(G)$ 与 $\zeta^{W_5^2}(G)$ 是 Kempe 极大平面图当且仅当

$$\left|L^4(G)\right| \leq 2 \tag{16}$$

关于 Kempe 极大平面图更为深入的研究将在后续文章中给出，其中包括定理 11 的详细论证，猜想 4 的论证，以及 G 与 $\zeta^{W_5^2}(G)$ 的关系等的研究。



(a) W_4^1　(b) W_5^1　(c) W_4^2　(d) W_5^2　(e) W_6^2

图 14　5 个基本多米诺构形

6 结束语

众所周知，表征 Kempe 图的特征一直是图着色理论与算法研究中的难点与热点问题。目前虽然在此领域中发表了不少学术论文，但是给出一般色数为 k 的 Kempe 图的充分必要条件仍很困难，故目前学界主要对一些特殊图类的 Kempe 等价类展开研究，如正则图等。本系列文章主要对另一类图——极大平面图的 Kempe 等价类展开了研究。

本文的主要贡献是：(1)发现了导致两个 4-着色是 Kempe 等价的关键子图为 2-色耳，故对 2-色耳的特征进行了深入研究；(2) 引入 σ -特征图，清晰地刻画了图 G 中所有着色之间的关联关系，并对 σ -特征图的性质进行了深入的研究；(3)揭示了非 Kempe 图的 Kempe 等价类可分为树型，圈型和循环圈型，并指出这 3 种类型可同时存在于一个极大平面图的 4-着色集中；(4)研究了 Kempe 图的特征，给出了 Kempe 图的多米诺递推构造法，并猜想 $\zeta^{W_5^1}(G)$ 与 $\zeta^{W_5^2}(G)$ 是 Kempe 图的充分必要条件是在图 G 中可 4-色漏斗子图的个数 ≤ 2 。

在后续文章中将对 3 类非 Kempe 图的结构与特征进行深入研究，特别，将给出 Kempe 极大平面图的充分必要条件。

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Theory on Structure and Coloring of Maximal Planar Graphs

(4) σ-Operations and Kempe Equivalent Classes

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Abstract: Let G be a k-chromatic graph. G is called a Kempe graph if all k-colorings of G are Kempe equivalent. It is an unsolved and hard problem to characterize the properties of Kempe graphs with chromatic number ≥ 3. The Kempe equivalence of maximal planar graphs is addressed in this paper. Our contributions are as follows: (1) Observe and study a class of subgraphs, called 2-chromatic ears, which play a critical role in guaranteeing the Kempe equivalence between two 4-colorings; (2) Introduce and explore the properties of σ-characteristic graphs, which clearly characterize the relations of all 4-colorings of a graph; (3) Divide the Kempe equivalent classes of non-Kempe 4-chromatic maximal planar graphs into three classes, tree-type, cycle-type, and circular-cycle-type, and point out that all these three classes can exist simultaneously in the set of 4-colorings of one maximal planar graph; (4) Study the characteristics of Kempe maximal planar graphs, introduce a recursive domino method to construct such graphs, and propose two conjectures.

Key words: Kempe maximal planar graph; Kempe transformation; σ- operation, Kempe equivalent class; σ-characteristic graph; 2-chromatic ear

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1 Introduction

The importance of planar graphs is reflected in two factors: mathematical theory and practical applications. At a theoretical level, there are many famous conjectures that have significant influences in graph theory and even in mathematics, such as the Four-Color Conjecture, the Uniquely Four-Colorable Planar Graph Conjecture, the Nine-Color Conjecture, and Three-colorable Problem *etc.*[1]. At a practical level, the planar graph theory can be directly applied to the fields of science such as circuit wiring[2] and information science[3].

Maximal planar graphs are an important subclass of planar graphs. Each face of a maximal planar graph is a triangle, so it is also called a triangulation. Since the researching objects of the Four-Color Conjecture can be confined to maximal planar graphs, numerous topics around maximal planar graphs have attracted the attentions and imaginations of researchers for more than a century, including degree sequences, constructions, coloring characters, traversability, and generating operations[4]. In the studying process of tackling Four-Color Conjecture, scholars proposed many conjectures, such as Uniquely Four-Colorable Planar Graph Conjecture and Nine-Color Conjecture, which gradually form the central research field on coloring theory of maximal planar graphs.

In terms of graph coloring theory, Kempe changes are proved to be one of the basic and most powerful tools. A Kempe change is to exchange two colors of a connected component of a 2-coloring induced subgraph, and remain the colors of the other vertices unchanged in a coloring graph. The σ-operation defined in this paper is the operation that contains one or more Kempe changes under a 4-coloring (The specific definition is in Section 3). A σ-operation is in fact a coloring-derived operation, which can induce a 4-coloring from another 4-coloring of a graph G.

Two k-colorings f and f' of a $k(\geq 2)$-chromatic graph G are called Kempe equivalent,

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if f' can be obtained from f by a sequence of Kempe changes. Obviously, this is an equivalent relation, by which we can classify all colorings of G. A Kempe equivalent class of G is the set of all the colorings that are mutually Kempe equivalent. Especially, a 4-colorable maximal planar graph is called a Kempe graph if any pair of 4-colorings of G is Kempe equivalent.

The Kempe changes are introduced by Kempe in his false proof of the four color theorem[5], after which Fisk first systematically studied this topic until 1977, showing that all 4-colorings of a maximal planar graph G are Kempe equivalent if G has no vertex with odd degree[6]. Subsequently, Meyniel[7], in 1978, showed that all 5-colorings of a planar graph are Kempe equivalent. In 1981, Vergnas and Meyniel proved: all 5-colorings of any simple graph not contractible to K_5 are Kempe equivalent[8]. In 2007, Mohar[9] proved that all k-colorings of a planar graph with chromatic number less than k are Kempe equivalent, and proposed the following conjecture.

Conjecture 1[9] For $k \geq 3$, all k-colorings of connected k-regular graphs that are not complete are Kempe equivalent.

In 2015, Feghali, *et al.*[10] have addressed the case when $k=3$ by showing that all 3-colorings of a connected cubic graph G are Kempe equivalent unless G is the complete graph K_4 or the triangular prism. Additionally, we1) prove that the conjecture is true when $k=4$. Conjecture 1 is open for $k \geq 5$.

In 2008, Cereceda, *et al.*[11] studied the k-chromatic characteristic graph $P_k(G)$ of a planar graph G. The vertex set of $P_k(G)$ consists of all k-colorings of G, and two k-colorings are joined by an edge in $P_k(G)$ if they differ in color on just one vertex of G. They proved that if G has chromatic number $k \in \{2,3\}$, then $P_k(G)$ is not connected, on the other hand, for $k \geq 4$ there are graphs with chromatic number k for which $P_k(G)$ is not connected, and there are k-chromatic graphs for which $P_k(G)$ is connected.

Some scholars have also considered Kempe changes and Kempe equivalent classes on edge coloring, such as McDonald and Mohar[12], Sarah and Ruth[13], and so on.

The complexity of graph algorithms has been investigated based on Kempe changes and Kempe equivalent classes. The interested readers can refer to Refs. [14–24].

This series of articles aim to establish the coloring operation system of maximal planar graphs, which contains two coloring-derived operations, one is the σ-operation, and the other is the τ-operation (also called the pseudo-edge-induced coloring method, which will be introduced and researched in another paper). For a given maximal planar graph G, σ-operations are very likely not able to induce all 4-colorings (or some desired one) of G from a given 4-coloring, while τ-operations can do well.

All graphs considered in this paper are finite, simple, and undirected. For a given graph G, we use $V(G)$, $E(G)$, $d_G(v)$, and $N_G(v)$ to denote the vertex set, the edge set, the degree of v, and the neighborhood of v in G (the set of all vertices adjacent to v) respectively, written as V, E, $d(v)$, and $N(v)$ for short if no confusion. The cardinality of the set $V(G)$ is denoted as $|V(G)|$, called the order of G. For a graph $H=(V',E')$, if $V' \subseteq V$, $E' \subseteq E$, and two ends of each edge in E' belong to V', then we call H a subgraph of G. And in the subgraph H, for $\forall u,v \in V(H)$, u,v are adjacent in G, if and only if they are also adjacent in H, then H is called an induced subgraph of G induced by V', denoted by $G[V']$. The join of two disjoint graphs G and H is the result of joining each vertex of G with every vertex of H, denoted by $G \vee H$. K_n is a complete graph having n vertices. A wheel is the join of a cycle C_n of n vertices and a trivial graph K_1, and denoted by W_n, where C_n and K_1 are called the wheel-cycle and the wheel-center of W_n, respectively. Also, we write the wheel-cycle C_n of W_n as C^x, where $V(K_1)=\{x\}$.

A vertex coloring f of a graph G, or simply a coloring, is an assignment from a color set to its vertex set. The coloring f is proper if any two adjacent vertices are assigned the different colors.

1)Liu Xiaoqing, Xu Jin, submitted to Dis. Math.

Unless special statement, any vertex coloring mentioned here is proper. A proper k-vertex coloring of a graph G, a k-vertex coloring or a k-coloringfor short, is a mapping f from the vertex set V to the color set $C(k)=\{1,2,\cdots,k\}$ such that $f(x)\neq f(y)$ for any $xy\in E(G)$. A graph G is k-colorable if it has a proper k-vertex coloring. The minimum k for which a graph G is k-colorable is called its chromatic number, denoted by $\chi(G)$. If $\chi(G)=k$, G is called a k-chromatic graph. Alternatively, each k-coloring f of G can be viewed as a partition $\{V_1,V_2,\cdots,V_k\}$ of V, where V_i denotes the set of all vertices assigned color i, called a color class of f. So it can be written as $f=(V_1,V_2,\cdots,V_k)$. In other words,

$$V(G)=\bigcup_{i=1}^{k}V_i,\ V_i\neq\phi,\ V_i\cap V_j=\phi,\ i\neq j,\ i,j=1,2,\cdots,k$$

where V_i is an independent set of G, $i=1,2,\cdots,k$. The set of all k-coloring of a graph G can be denoted by $C_k(G)$. For a k-chromatic graph G, we use the notation $C_k^0(G)$ to denote the set of all the partitions of vertex set of G, where each partition is corresponding to a k-coloring, and we call $C_k^0(G)$ the set of partitions of k color classes of G.

Let G be a k-colorable graph, and f be a k-coloring of G, where the color set is $\{1,2,\cdots,k\}$. The subgraph induced by vertices with color i or t under f, denoted by G_{it}^f, is referred to as a 2-coloring induced subgraph of G, where $i,\ t=1,2,\cdots,k, i\neq t$. When there is no scope for ambiguity, we use G_{it} instead of G_{it}^f; A component of G_{it} is called an it-component.

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing is called a planar embedding of the graph. Any planar graph considered in the paper is assumed one of its planar embeddings. A maximal planar graph is a planar graph to which no new edges can be added without violating planarity. A triangulation is a planar graph in which every face is bounded by three edges (including its infinite face). It can be easily proved that maximal planar graphs are triangulations, and vice versa.

The graph shown in Fig. 1 is called a funnel,

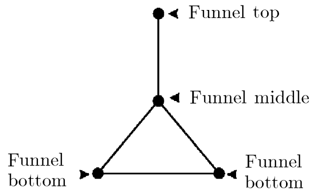


Fig. 1 The funnel

where the 1-degree vertex is the funnel top, the 3-degree vertex is the funnel middle, and the two 2-degree vertices are the funnel bottoms. If an induced subgraph of a graph is a funnel, then it is called a funnel subgraph of the graph. For more details about funnels, please refer to this series of articles (2)[25].

Suppose that G is a 4-colorable maximal planar graph with $\delta\geq 4$, and L is a funnel subgraph of G. We define

$$f_L^4(G)=\left\{f: f\in C_4^0(G),\ \ |f(L)|=4\right\} \tag{1}$$

If $f_L^4(G)\neq\phi$, then L is called a potential 4-chromatic funnel of G, and the set of all potential 4-chromatic funnels is denoted by $L^4(G)$.

In this paper, we introduce the σ-operation, and find that the inner mechanism that a new 4-coloring can be derived from a given 4-coloring by σ-operations is closely related with a class of subgraphs, called 2-chromatic ears. Therefore, we make an in-depth research on 2-chromatic ears in Section 2. Section 3 introduces and explores the properties of σ-characteristic graphs, which clearly characterize the relations of all 4-colorings of a graph. Section 4 discusses the limitations of σ-operations, namely, some 4-colorings may not be able to be derived from a given 4-coloring by σ-operations, based on which we partition the Kempe equivalent classes of non-Kempe graphs into three classes: tree-type, cycle-type, and circular-cycle-type. Section 5 studies the characteristics of Kempe graphs, introduces a recursive domino method to construct Kempe graphs, and proposes two conjectures to describe the properties of Kempe maximal planar graphs.

For more notations and terminologies, we refer the readers to Refs. [25–27].

2 2-Chromatic Ears

In this section, we first give the definition and classification of 2-chromatic cycles, and then we introduce the 2-chromatic ears, which is the root of implementing σ-operations continuously.

2.1 Related definitions of 2-chromatic cycles

Let G be a 4-colorable maximal planar graph, H a subgraph of G, and f a 4-coloring of G. We use $f(H)$ to denote the set of all the colors assigned to $V(H)$ under f. The sets of all the interior vertices and exterior vertices of any cycle C of G are denoted by V_C^{in} and V_C^{out}, respectively.

Suppose that C is an even-cycle of G, $f \in C_4^0(G)$. If $|f(C)|=2$, then we call C a 2-chromatic cycle of f, and also say that f contains the 2-chromatic cycle C. The two colors in $f(C)$ are called cycle-colors, and the other two colors are called non-cycle colors. For a 2-chromatic cycle C of f, let u,v be two different vertices in C. If $uv \in E(G)$ and uv is in the interior of C, then C is called a 2-chromatic chord-cycle of f, and the edge uv is called a chord of C; If there is a path P $(\neq uv)$ colored with $f(C)$ in the interior of C that connects u and v, then C is referred to as a 2-chromatic chord-path cycle of f, and P is called a chord-path of C; If C is neither a 2-chromatic chord-cycle nor a 2-chromatic chord-path cycle, then we call C a-chromatic basic cycle.

For example, the cycle C shown in Fig. 2(a) is a chord-cycle with a chord v_1y_5; the cycle C shown in Fig. 2(b) is a chord-path cycle with a chord-path $v_1v_2v_3v_4v_5$. C_1 and C_2 both shown in Figs. 2(a) and 2(b) are 2-chromatic basic cycles.

For a 4-colorable maximal planar graph G under a 4-coloring, and a given 2-chromatic cycle C of G, which type the 2-chromatic cycle C belongs to depends on the ways of planar embedding of G. As the graph shown in Fig. 2(a), if its planar embedding is converted to the graph shown in Fig. 2(c), then the 2-chromatic cycle C_1 is a 2-chromatic chord-path cycle with a chord-path $v_1x_1x_2x_3x_4x_5v_5y_5$, while C and C_2 are the 2-chromatic basic cycles. Unless special declaration, all 2-chromatic cycles in the following argument are 2-chromatic basic cycles. For a coloring $f \in C_4^0(G)$, we denote by $C^2(f)$ the set of all 2-chromatic cycles under f. A cycle C of G is 2-colorable if there exists a coloring $f \in C_4^0(G)$ such that $|f(C)|=2$. Use $C^2(G)$ to denote the set of all 2-colorable cycles of G. It is easy to show that

$$C^2(G) = \bigcup_{f \in C_4^0(G)} C^2(f) \tag{2}$$

Let G be a 4-colorable maximal planar graph with $\delta(G) \geq 4$, f a 4-coloring of G, and C_1, C_2 two 2-chromatic cycles of f. We say that C_1 and C_2 are joint if the following two conditions hold:

(1) $|f(C_1) \cap f(C_2)| = 1$;

(2) $V_{C_1}^{\text{in}} \cap V(C_2) \neq \phi$, $V_{C_1}^{\text{out}} \cap V(C_2) \neq \phi$.

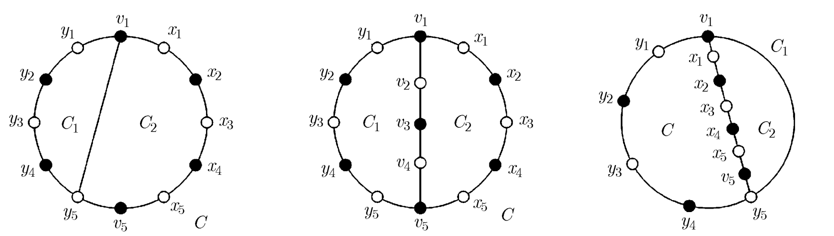
Otherwise, C_1 and C_2 are disjoint.

Suppose $C^2(G) = \{C_1, C_2, \cdots, C_m\}$ with $m \geq 2$. For any two 2-colorable cycles $C_1, C_t \in C^2(G)$, if there exist a sequence of 2-colorable cycles $C_1, C_2, \cdots, C_t$ with the corresponding 4-colorings $f_1, f_2, \cdots, f_t$, such that C_i and C_{i+1} $(1 \leq i \leq t-1)$ are intersected under f_i, then we say that C_1 and C_t are relevant, otherwise, C_1 and C_t are irrelevant.

For example, in Fig. 3, C_1, C_2, C_3, C_4 are four 2-colorable cycles of $C^2(G)$; f_1, f_2 are two colorings of G, where C_1, C_2, C_3 are three successively intersected 2-chromatic cycles of f_1; C_3, C_4 are two intersected 2-chromatic cycles of f_2. Therefore, it is not hard to see that C_1 and C_4 are relevant.

2.2 Related definitions and characteristics of 2-chromatic Ears

Suppose that G is a 4-colorable maximal



(a) A 2-chromatic chord cycle (b) A 2-chromatic chord-path cycle (c) A 2-chromatic basic cycle

Fig. 2 Diagrams of the chord cycle, the chord-path cycle and the basic cycle

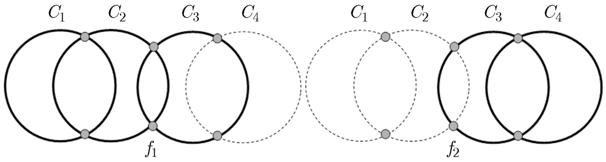


Fig. 3 An example of two relevant2-chromatic cycles C_1, C_4

planar graph, $f \in C_4^0(G)$, $C \in C^2(f)$, $f(C)=\{1,2\}$, and x,y are two different vertices in C with the same color under f. We use $P(x,y)$ to denote a path between x and y with $|V(P(x,y))| \geq 3$. If $f(P(x,y)) \not\subset \{1,2\}$ and $|f(P(x,y))| = 2$, then we call $P(x,y)$ a 2-chromatic ear of C (or simply an ear of C), and call x,y the ear roots of $P(x,y)$. We write $Ed(C)$ for the set of all ears of C. The ears of $Ed(C)$ are divided into two classes, inner ear which is in the interior of C, and outer ear which is in the exterior of C. If $f(P(x,y)) = \{1,3\}$, then $P(x,y)$ is called 13-ear. Analogously, we have 14-ear, 23-ear, and 24-ear. Furthermore, $f(P) \setminus f(C)$ is called ear-edge color. Obviously, the ear-edge color is neither 1 nor 2.

Let C be a 2-chromatic cycle of a 4-coloring, and $P(x,y)$ and $P(x',y')$ be two ears of C. If $f(x) = f(x')$, then we say $P(x,y)$ and $P(x',y')$ to be homologous; if $x = x'$ and $y = y'$, then we say $P(x,y)$ and $P(x',y')$ to be co-rooted. Obviously, if two ears are co-rooted, then they must be homologous, but the converse may not be true.

Suppose that $P(x,y)$ and $P(x',y')$ are two ears of a 2-chromatic cycle C. We say $P(x,y)$ and $P(x',y')$ to be homochromatic if one of the following conditions hold:

(1) Both $P(x,y)$ and $P(x',y')$ are inner ears or outer ears, and $f(P(x,y)) = f(P(x',y'))$.

(2) $P(x,y)$ and $P(x',y')$ are homologous ears with different ear-edge colors, and one is an inner ear and the other is an outer ear.

Otherwise, we say $P(x,y)$ and $P(x',y')$ to be heterochromous ears.

Suppose that $P_1, P_2, \cdots, P_m (m \geq 2)$ are ears of a 2-chromatic cycle C. We call them to be path-connected if $P_1 \cup P_2 \cup \cdots \cup P_m$ is a path of G, and to be cycle-connected if $P_1 \cup P_2 \cup \cdots \cup P_m \triangleq C'$ is an even-cycle of G. Let $Q(C)$ be the set of cycles containing only ears of C. A cycle is referred to as an inner ear-cycle (resp. outer ear-cycle) if it contains only inner (resp. outer) ears. In addition, if a cycle contains both inner ear-cycles and outer ear-cycles, then we call the cycle a mixed ear-cycle. We write $Q^i(C)$, $Q^e(C)$, and $Q^m(C)$ for the set of inner ear-cycles, outer ear-cycles, and mixed ear-cycles, respectively. It is not difficult to see that

$$Q(C) = Q^i(C) \cup Q^e(C) \cup Q^m(C) \qquad (3)$$

Fig. 4 contains 5 ears P_1, P_2, P_3, P_4, and P_5 of a 2-chromatic cycle C. P_1, P_2, P_3, P_4 are mutually homologous; P_1 and P_2 are co-rooted; P_1, P_3, P_5 are inner ears; P_2 and P_4 are outer ears. P_1, P_3, P_4 (resp. P_2, P_3, P_4) are path-connected; P_1, P_2 are cycle-connected.

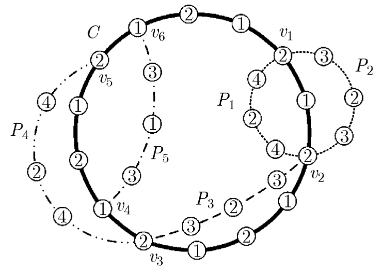


Fig. 4 The diagram of ears, co-root ears, and homologous ears

Theorem 1 Suppose that G is a 4-colorable maximal planar graph with $\delta(G) \geq 4$, $f \in C_4^0(G)$. If f contains only one 2-chromatic cycle C, say $f(C) = \{1,2\}$, then

(1)Every cycle in $Q^i(C)$ and $Q^e(C)$ contains some heterochromous ears of C;

(2)The operation of interchanging the colors of all 34-components in the interior of C can induce a new coloring f^c such that f^c contains a 2-chromatic cycle C' different from C if and only if

$$C' \in Q^m(C) \qquad (4)$$

and any pair of ears of C' are homochromatic.

Proof (1) Suppose that there exist a cycle C' in $Q^i(C)$ or $Q^e(C)$, which is consisted of only homochromatic ears. Then, by the definition of homochromatic ears, we have that any pair of ears of C' not only are homologous, but also have the same color set under f. Therefore, C' is a 2-chromatic cycle of f. This contradicts the assumption of C.

(2)Necessity. As C is the unique 2-chromatic cycle of f, it has that $C' \in Q^m(C)$. Notice that any pair of ears of C' are homologous. If there are two heterochromous ears P, P' in C', then P, P' are two inner ears, or two outer ears, or one of P, P' is an inner ear and the other is an outer ear of C. However, in any case we can see that C' is by no means a 2-chromatic cycle of f^c, and a contradiction.

Sufficiency. $C' \in Q^m(C)$ implies that C' is a cycle, and any pair of ears of C' are homologous. Suppose that P, P' are a pair of ears of C'. Obviously, P, P' are homochromatic. When P, P' are two inner ears (or outer ears), then $f(P) = f(P')$; when one of P, P' is an inner ear and the other is an outer ear, then $|f(P) \cup f(P')| = 3$, and $\{1,2\} \not\subset f(P) \cup f(P')$. Therefore, $P \cup P'$ are colored with exactly two colors under f^c, *i.e.* C' is a 2-chromatic cycle of f^c. This completes the proof of the theorem.

Suppose that G is a 4-colorable maximal planar graph, $f \in C_4^0(G)$, and C is the unique 2-chromatic cycle of f, $f(C) = \{1,2\}$. Let f^c be a new coloring obtained from f by the operation of interchanging the colors of all 34-components in the interior of C. If f^c contains at least two 2-chromatic cycles, then any 2-chromatic cycle C' ($\neq C$) of f^c is consist of only homochromatic ears, as described in the following.

(1) If C' has exactly two homochromatic ears, then one is an inner ear, and the other is an outer ear, shown in Fig. 5(a);

(2) If C' has three homochromatic ears, then there are two possible structures shown in Fig. 5(b) and 5(c), respectively;

(3) If C' has four homochromatic ears, then its structure may be the one shown in Fig. 5(d) or Fig. 5(e);

(4) If C' contains five or more homochromatic ears, then its general structure is shown in Fig. 5(f).

3 The σ- Operation and σ- Characteristic Graphs

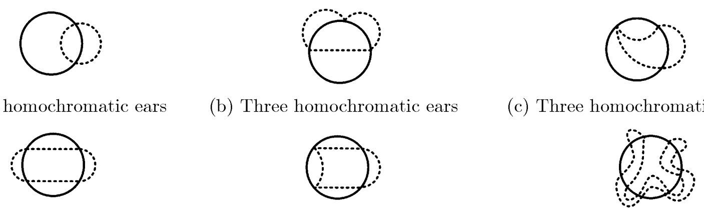
XU introduced the tree-coloring and cycle-coloring in Ref. [26]. For the convenience of description, we restate them here. Let G be a 4-colorable maximal planar graph with color set $\{1,2,3,4\}$, $f \in C_4^0(G)$. If $\exists i,t \in \{1,2,3,4\}$ such that G_{it}^f has a cycle, then f is referred to as a cycle-coloring, and we call G is cycle-colorable. Conversely, if $\forall i,t \in \{1,2,3,4\}$, G_{it}^f contains no cycle, then f is referred to as a tree-coloring of G, and G is tree-colorable. If G is tree-colorable (cycle-colorable) but not cycle-colorable (tree-colorable), the G is also called purely tree-colorable (purely cycle-colorable). Furthermore, if G is both tree-colorable and cycle-colorable, then G is also called mixed colorable. Obviously, for any 4-colorable maximal planar graph G, colorings in $C_4^0(G)$ can be partitioned into two classes: cycle-colorings and tree-colorings.

3.1 The σ- operation and Kempe equivalent classes

Let G be a 4-colorable maximal planar graph with color set $\{1,2,3,4\}$, and $f \in C_4^0(G)$. Suppose C is a 2-chromatic cycle of f such that $f(C) = \{1,2\}$. A σ-operation respect to C of f, denoted by $\sigma(f,C)$ (abbreviated as $\sigma(f)$ if there is no confusion), is the operation of interchanging the colors of all the 34-components inside C. Obviously, a σ-operation is a kind of coloring-derived operations, by which a new cycle-coloring (denoted by f^c) can be obtained from a given cycle-coloring f, namely

$$\sigma(f,C) = f^c \tag{5}$$

We say that f^c and f are complementary respect to C. Equation (5) can also be represented as $\sigma(f,C) = f^c$ if there is no confusion. It is easy to see that, when there is only one 34-component inside C,



(a) Two homochromatic ears　(b) Three homochromatic ears　(c) Three homochromatic ears
(d) Four homochromatic ears　(e) Four homochromatic ears　(f) More than five homochromatic ears

Fig. 5　Possible structures of cycle-connected and homochromatic ears, where C is indicated by a solid line then a σ-operation is equal to a Kempe change; when there are $m(\geq 2)$ 34-components inside C, then a σ-operation contains m Kempe changes. Therefore, f^c and f are Kempe equivalent.

Let $f_0 \in C_4^0(G)$, we call

$$F^{f_0}(G) \triangleq \left\{f; f \text{ and } f_0 \text{ are Kempe equivalent}, f \in C_4^0(G)\right\} \quad (6)$$

a Kempe equivalent class of f_0.

An 11-order maximal planar graph shown in Fig. 6 has totally eight 4-colorings, denoted by $f_1 \sim f_8$, respectively. For f_1, f_2, f_3, f_4, it is not difficult to see $\sigma(f_1,C_1)=f_2$; $\sigma(f_2,C_2)=f_3$; $\sigma(f_2,C_3)=f_4$. Moreover, G has seven 2-colorable cycles $C_1 \sim C_7$ in total, thus $|C^2(G)|=7$.

The following is a straight forward result by the definition of σ-operations:

Theorem 2 Suppose f is a 4-coloring of G, and C is a 2-chromatic cycle of f. Then

$$\sigma\left(\sigma(f,C),C\right)=f \quad (7)$$

The aim of the σ-operations is to derive the Kempe equivalent class $F^f(G)$ of f in $C_4^0(G)$ by a sequence of σ-operations starting with a 4-coloring f. It has been known that, all 5-colorings of a planar graph are Kempe equivalent. But the situation is different for $C_4^0(G)$, which will be discussed later.

3.2 Definition of σ- characteristic graphs

In this section, we introduce σ-characteristic graphs for σ-operations, which clearly characterize the relations of all 4-colorings of G.

Let G be a 4-colorable maximal planar graph, and $C_4^0(G)=\{f_1,f_2,\cdots,f_n\}$. The σ-characteristic graph of G, denoted by G_4^σ, is a graph with the vertex set $\{f_1,f_2,\cdots,f_n\}$ such that two vertices f_i and f_t are adjacent in G_4^σ if and only if they are complementary respect to some 2-chromatic cycle C in G for $i,t=1,2,\cdots,n,\ i\neq t$. By this definition, we can view a σ-characteristic graph as an edge-labelled graph, in which the label appearing on an edge is a 2-chromatic cycle that results in two complementary colorings (the two ends of the edge). When we only concern the topological structure of a σ-characteristic graph, the labels in G_4^σ can be deleted.

The graph G shown in Fig. 6 has totally eight 4-colorings. Its σ-characteristic graph and corresponding topological structure are shown in Figs. 7(a) and 7(b). It is evident that G_4^σ is connected and is a tree. Therefore, we can obtain all 4-colorings of G by σ-operations based on any given 4-coloring. In addition, consider the icosahedron; it has totally ten 4-colorings, each of which is a tree-coloring. So its σ-characteristic graph is an empty graph of order 10.

3.3 Basic properties of σ-characteristic graphs

Let G be a 4-colorable maximal planar graph. If $|C_4^0(G)|=1$, then we call G a uniquely four-colorable maximal planar graph.

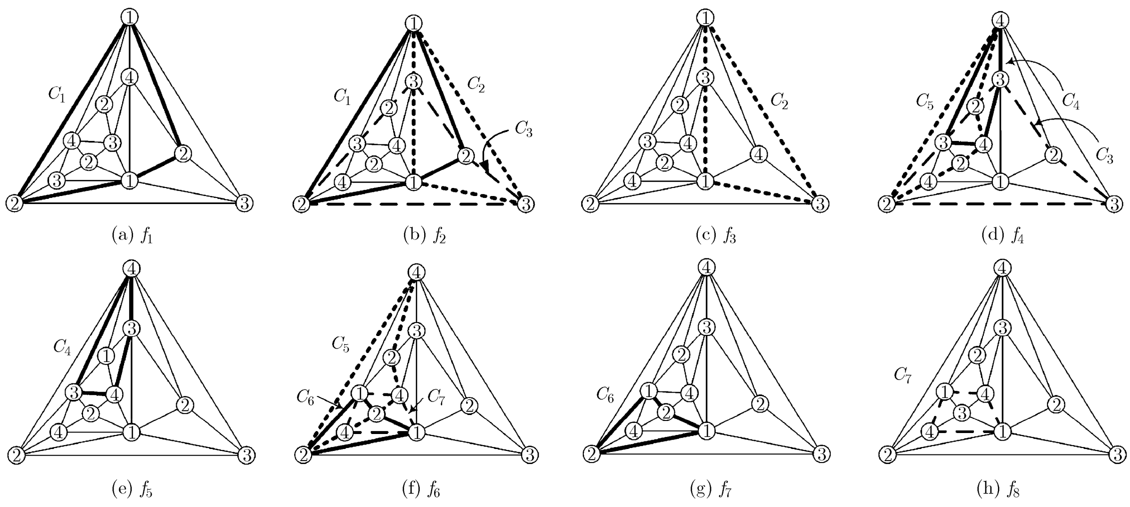
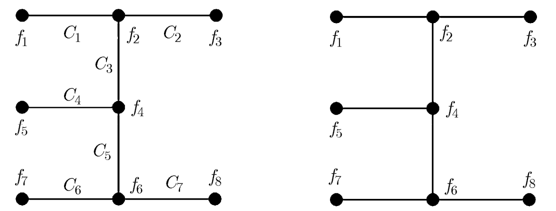


Fig. 6 Examples of complementary colorings



(a) The σ-characteristic graph (b) The topological structure

Fig. 7 The σ- characteristic graph and its topological structure of the graph shown in Fig. 6

Theorem 3 For a 4-colorable maximal planer graph G, let G_4^σ be the σ-characteristic graph of G. We have

(1) G is a uniquely four-colorable maximal planar graph if and only if $G_4^\sigma \cong K_1$;

(2) For any $f \in V(G_4^\sigma)$, $d_{G_4^\sigma}(f) = k$ if and only if f contains k 2-chromatic cycles;

(3) If there exists a $f \in C_4^0(G)$, which contains $k(\geq 1)$ 2-chromatic cycles, *i.e.* $d_{G_4^\sigma}(f) = k$, then

$$\left|C_4^0(G)\right| \geq k+1 \quad (8)$$

Among all maximal planar graphs with order at most 11, tree-colorings account for about 2 percent of all 4-colorings[26]. We conjecture that the number of tree-colorings is much less than that of cycle-colorings of maximal planar graphs with $\delta \geq 4$. Hence, for a given cycle-coloring f of a 4-chromatic maximal planar graph G, it is possible to derive all 4-colorings of $C_4^0(G)$ from f by means of the σ-operations.

Theorem 4 Let G be a 4-colorable maximal planar graph. If G_4^σ is connected, then the running time of algorithms generating all 4-colorings is greater than that of algorithms generating one 4-coloring by only $\left|V(G_4^\sigma)\right| - 1$.

According to Theorem 4, if G is a 4-chromatic Kempe maximal planar graph, then the algorithm finding all 4-colorings is polynomial time equivalent to the algorithm finding one 4-coloring.

Obviously, G_4^σ and $C_4^0(G)$ are tightly related with each other. Thus it can be seen that exploring the structure of G_4^σ is an essential work for investigating $C_4^0(G)$. To do this, we will study each Kempe equivalent class $F^f(G)$ of $f \in C_4^0(G)$.

Theorem 5 Let G be a 4-colorable maximal planar graph with $\delta(G) \geq 4$, and $f \in C_4^0(G)$. Then, G_4^σ contains no triangles.

Proof By contradiction. Suppose that G_4^σ contains a triangle $f_1 f_2 f_3 f_1$. Without loss of generality, we assume

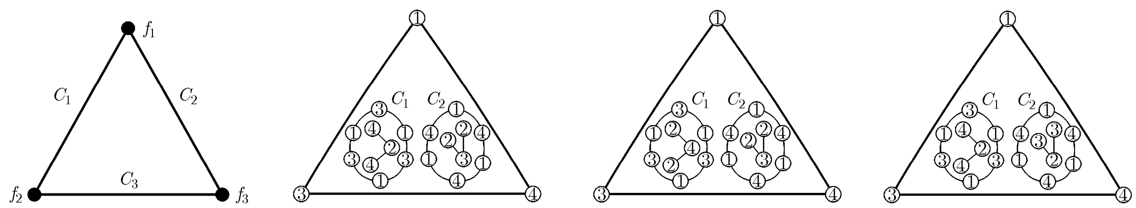
$$\sigma(f_1, C_1) = f_2,\ \sigma(f_1, C_2) = f_3,\ \sigma(f_2, C_3) = f_3 \quad (9)$$

Then, $C_1, C_2 \in C^2(f_1)$, $C_1, C_3 \in C^2(f_2)$, $C_2, C_3 \in C^2(f_3)$, as shown in Fig. 8(a). Let $C(4) = \{1,2,3,4\}$ be the color set. Denote by $V_{C_i}^{*\text{in}}$ the set of vertices in the interior of C_i and with color not appearing on C_i, for $i = 1,2$. We need to consider two cases as follows.

Case 1 C_1, C_2 are disjoint under the coloring f_1; see Fig. 8(b). Here, colors assigned to C_1 and C_2 can be the same or not, but the proofs are analogously.

Obviously, $\forall u \in V_{C_1}^{*\text{in}}$, we have $f_1(u) \neq f_2(u)$; and $\forall v \in V(G) \setminus V_{C_1}^{*\text{in}}$, we have $f_1(v) = f_2(v)$ (see Figs. 8(b) and 8(c)). Similarly, $\forall u \in V_{C_2}^{*\text{in}}$, it has that $f_1(u) \neq f_3(u)$, and $\forall v \in V(G) \setminus V_{C_2}^{*\text{in}}$, $f_1(v) = f_3(v)$ (see Figs. 8(b) and 8(d)). Therefore, $\forall u \in V_{C_1}^{*\text{in}} \cup V_{C_2}^{*\text{in}}$, $f_2(u) \neq f_3(u)$, and $\forall v \in V(G) \setminus V_{C_1}^{*\text{in}} \cup V_{C_2}^{*\text{in}}$, $f_2(v) = f_3(v)$. Thus, there does not exist a 2-chromatic cycle C_3 satisfying the condition $V_{C_3}^{*\text{in}} = V_{C_1}^{*\text{in}} \cup V_{C_2}^{*\text{in}}$ for f_2, f_3, and a contradiction.

Case 2 C_1, C_2 are joint under the coloring f_1. Then $f_1(C_1) \neq f_1(C_2)$. Without loss of generality, we assume $f_1(C_1) = \{1,2\}$ and $f_1(C_2) = \{1,3\}$ (see Fig. 9(a)). Let $u \in V_{C_1}^{*\text{in}} \cap V(C_2)$ and $V_{C_2}^{*\text{in}} \cap V(C_1)$. Then it has that $f_1(u) = 3$, $f_1(v) = 2$; $f_2(u) = 4$, $f_2(v) = 2$; $f_3(u) = 3$, $f_3(v) = 4$. Clearly, f_2 does



(a) A subgraph of G_4^σ (b) f_1 (c) f_2 (d) f_3

Fig. 8 The first case of the proof of the Theorem 5: C_1, C_2 are disjoint

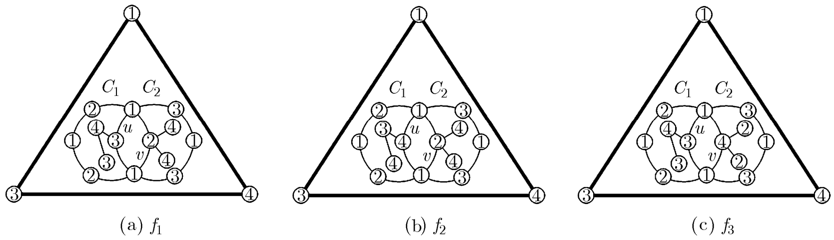


Fig. 9 The second case of the proof of the Theorem 5: C_1, C_2 are joint.

not contain any 2-chromatic cycle C_3, such that f_2 and f_3 are two complementary colorings respect to C_3; a contradiction.

This completes the proof of Theorem 5.

By Theorem 5, it has that the σ-characteristic graph G_4^σ of a 4-colorable maximal planar graph G with $\delta \geq 4$ is triangle-free, which means that the induced subgraph $G_4^\sigma\left[\overline{N^f(G)}\right]$ is a star, where $\overline{N^f(G)}$ is the closed neighborhood of vertex f in G_4^σ. So, we have,

Corollary 1 Let G be a 4-colorable maximal planar graph with $\delta \geq 4$. Then, for any cycle-coloring $f \in C_4^0(G)$, the induced graph $G_4^\sigma\left[\overline{N^f(G)}\right]$ is a star.

Theorem 6 Let G be a 4-colorable maximal planar graph, and $f \in C_4^0(G)$. Suppose that C is a 2-chromatic cycle of f, and $d_{G_4^\sigma}(f)=1$. If $\sigma(f) = f^c$ has degree at least 2 in G_4^σ, then G has a 4-coloring $f' \in C_4^0(G)$ such that $|f'(C)| \geq 3$.

Proof Since $|C^2(f)|=1$, we take a 2-chromatic cycle C of f, and assume $f(C)=\{1,2\}$ (see Fig. 10(a)). Because f^c contains at least two 2-chromatic cycles, thus there exist at least a pair of vertices of C assigned the same color under f, say v_1, v_2, based on which there are a 14-ear outside C and a 13-ear inside C (see Fig. 10(a)), where we assume that $f(v_1)=f(v_2)=1$. Now, we can obtain a 4-coloring f^c by implementing a σ-operation respect to C. Notice that f^c contains at least two 2-chromatic cycles C and C' shown in Fig. 10(b). Therefore, we can obtain a new 4-coloring f' by implementing a σ-operation respect to C', and $|f'(C)|=3$ (see Fig. 10(c)). Hence, the conclusion holds.

Theorem 7 Suppose that G is a 4-colorable maximal planar graph with $\delta(G) \geq 4$. Then, G_4^σ does not contain any pair of vertices with degree 1 and 2, respectively.

Proof To the contrary, suppose that f_1, f_2 are two adjacent vertices in G_4^σ such that $d_{G_4^\sigma}(f_1)=1$ and $d_{G_4^\sigma}(f_2)=2$. Let C_1 be the unique 2-chromatic cycle of f_1, and let f_2 contain two joint 2-chromatic cycles C_1, C_2. Clearly, f_1 and f_2 are complementary respect to C_1. Without loss of generality, assume that $f_1(C_1)=\{1,2\}$, $f_2(C_1)=\{1,2\}$, $f_2(C_2)=\{1,3\}$. We use $G_1^{C_1}$, $G_2^{C_1}$ to denote the subgraphs of G induced by $C_1 \cup V_{C_1}^{\text{in}}$ and $C_1 \cup V_{C_1}^{\text{out}}$, respectively, and use r_{ij} to denote the number of ij-components in $G_1^{C_1}$, for $i,j \in \{1,2,3,4\}$ and $i \neq j$.

Since the graph G, under the coloring f_1, has a unique 2-chromatic cycle C_1, we have that G_{34} consists of two connected components, and G_{13}, G_{14}, G_{23}, G_{24} are connected. Now, we can see that the r_{14} 14-components of $G_1^{C_1}$ are connected to a

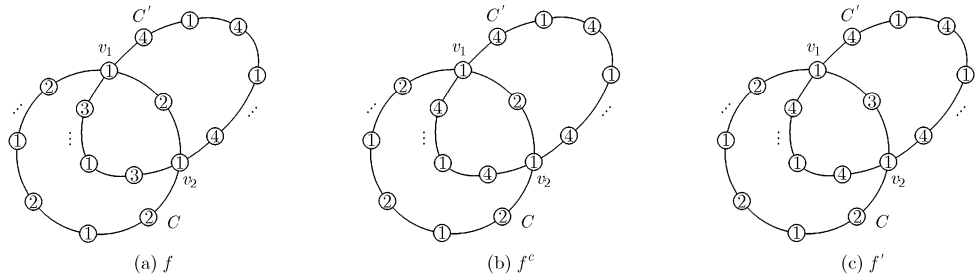


Fig. 10 The diagram for the proof of Theorem 6

component by the $(r_{14}-1)$ 14-paths of $G_2^{C_1}$; and the r_{13} 13-components of $G_1^{C_1}$ are connected into one component by the $(r_{13}-1)$ 13-paths of $G_2^{C_1}$.

Then, we can obtain a 4-coloring f_2 by exchanging the colors of 34-componentinside C_1. Under the coloring f_2, it is easy to see that the number of 13-components in $G_1^{C_1}$ is r_{14}, and $G_2^{C_1}$ still contains $(r_{13}-1)$ 13-paths with both the initial vertex and the terminus vertex on C_1. Because all of these 13-paths connect the r_{14} 13-components of $G_1^{C_1}$ into a component with a 13-cycle C_2, it has that $r_{13}>r_{14}$. In addition, the number of 14-components in $G_1^{C_1}$ is changed into r_{13}, and $G_2^{C_1}$ still contains $(r_{14}-1)$ 14-paths with both the initial vertex and the terminus vertex on C_1. Because $r_{13}>r_{14}$, it follows that the r_{13} 14-components in $G_1^{C_1}$ can not be connected into a component by the $(r_{14}-1)$ 14-paths. Hence, G_{14} is disconnected under the coloring f_2, *i.e.* G_{23} contains a 2-chromatic cycle, and a contradiction. Hence, the result holds.

The case of G_4^{σ} being disconnected is the key point of our research, which will be studied deeply in the following section.

4 Types of Kempe Equivalent Classes of Non-Kempe Graphs

Let G be a 4-colorable maximal planar graph, and $f, f' \in C_4^0(G)$. If f and f' are non-Kempe equivalent, then f' can not be obtained by implementing a series of σ-operations starting with f. The reason can be attributed to the 2-chromatic cycles of f: (1) $C^2(f)=\phi$, *i.e.* f is a tree-coloring; (2) $C^2(f)$ contains 2-chromatic unchanged-cycles (the definition will be defined later); (3) $C^2(f)$ contains the circular 2-chromatic cycle (the definition will be defined later). Based on these three cases, we divide Kempe equivalent classes of non-Kempe graphs into three classes: tree-type, cycle-type, and circular-cycle-type. In the following, we will introduce these three types, respectively. For convenience, we use the notation $\omega(G_4^{\sigma})$ to denote the number of Kempe equivalent class of G.

4.1 Tree-type Kempe equivalent class

If f is a tree-coloring of a 4-colorable maximal planar graph G, then, $C^2(f)=\phi$, namely, all six 2-colored induced subgraphs are connected. Therefore, $F^f(G)=\{f\}$. We refer to this Kempe equivalent class as a tree-type Kempe equivalent class of non-Kempe graph G, and call G a tree- type maximal planar graph. Thus, for any $f' \in C_4^0(G)$ and $f' \neq f$, we have

$$f' \notin F^f(G) \tag{10}$$

It follows that

Theorem 8 Suppose that G is a non-uniquely 4-colorable maximal planar graph. Then, $\omega(G_4^{\sigma}) \geq 2$; If G is purely tree-colorable, then $\omega(G_4^{\sigma}) = |C_4^0(G)|$. If G is mixed colorable and $C_4^0(G)$ contains t tree-colorings, then $\omega(G_4^{\sigma}) \geq t+1$.

4.2 Cycle-type Kempe equivalent classes

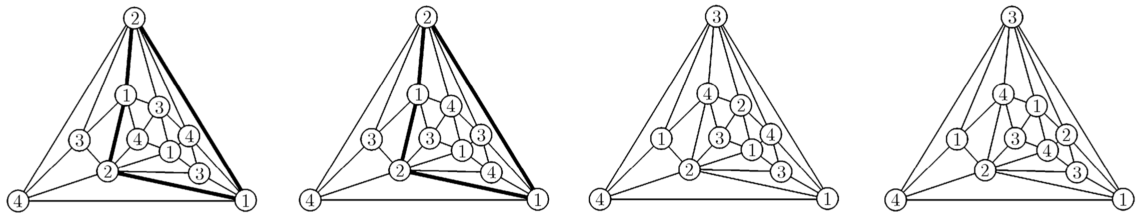
Suppose that G is a 4-colorable maximal planar graph, $f \in C_4^0(G)$, and $C \in C^2(f)$. If $\forall f' \in F^f(G)$, $|f'(C)|=2$, then we call C a 2-chromatic unchanged-cycle of f, f a 2-chromatic unchanged-cycle coloring of G, and G a cycle-type maximal planar graph based on C. By this definition, if we denote by f^c the resulting coloring obtained by implementing σ-operations respect to C under f, then we have,

$$C^2(f)=C^2(f^c) \tag{11}$$

In Eq. (11), we do not consider the colors on 2-chromatic cycles in $C^2(f)$ and $C^2(f^c)$. That is to say, for a 2-chromatic cycle C both in $C^2(f)$ and $C^2(f^c)$, $f(C)$ may be different from $f^c(C)$.

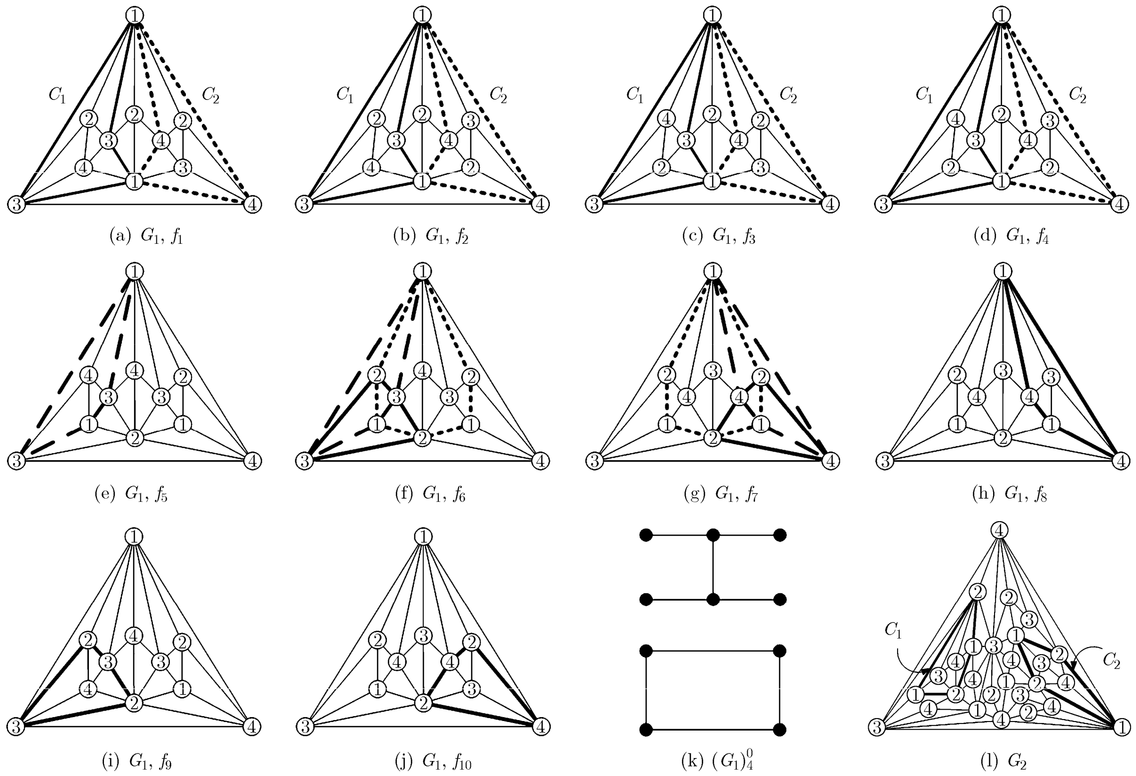
Obviously, the cycle-coloring f^c satisfying Eq. (11) is also a 2-chromatic unchanged-cycle coloring of G. The cycle-type maximal planar graph shown in Fig. 11 has totally four 4-colorings f_1, f_2, f_3 and f_4, where f_1 and f_2 are two complementary 2-chromatic unchanged-cycle colorings respect to C, and the unique 2- chromatic unchanged-cycle of them is 12-cycle.

Fig. 12 exhibits two cycle-type maximal planar graphs G_1 and G_2, each of which contains two 2-chromatic unchanged-cycles, where two 2-chromatic unchanged-cycles in G_1 have two common vertices, and G_2 has two disjoint 2-chromatic unchanged-cycles (see Fig. 12(l), marked by thick lines). Figs. 12(a)~12(j) exhibit all the 4-colorings of G_1, and the corresponding σ-



(a) f_1: cycle-coloring,12-cycle (b) f_2: cycle-coloring, 12-cycle (c) f_3: tree-coloring (d) f_4: tree-coloring

Fig. 11 An example of cycle-type maximal planar graph containing only one 2-chromatic unchanged-cycle



(a) G_1, f_1 (b) G_1, f_2 (c) G_1, f_3 (d) G_1, f_4 (e) G_1, f_5 (f) G_1, f_6 (g) G_1, f_7 (h) G_1, f_8 (i) G_1, f_9 (j) G_1, f_{10} (k) $(G_1)_4^0$ (l) G_2

Fig. 12 Two maximal planar graphs containing two 2-chromatic unchanged-cycles

characteristic graph is shown in Fig. 12(k).

Let G be a 4-colorable maximal planar graph, $f \in C_4^0(G)$, and $C \in C^2(f)$. If $\exists f' \in C_4^0(G)$ satisfies

$$|f'(C)| \geq 3 \tag{12}$$

then we say C is breakable.

Theorem 9 Let G be a 4-colorable maximal planar graph with $\delta(G) \geq 4$, C_1 and C_t be two relevant 2-colorable cycles of G. Then both C_1 and C_t are breakable.

Proof The result follows from definitions of the 2-chromatic colorable-cycles and σ-operations.

For two 2-chromatic cycles with the condition of Theorem 9, it is easy to see that these two cycles are breakable. However, we can not break a 2-chromatic unchanged-cycle C of f by implementing a series of σ-operations starting with f, *i.e.* we can not obtain a $f' \in C_4^0(G)$ satisfying Eq. (12) through σ-operations and f. So, f and f' are not Kempe equivalent. The root reason why f and f' are not Kempe equivalent is the 2-chromatic unchanged-cycle. We refer to such equivalent class $F^f(G)$ as a cycle-type Kempe equivalent class. For a 4-coloring set containing exactly $k\ (\geq 2)$ 2-chromatic unchanged-cycles, the induced subgraph in G_4^σ induced by this set is a k-dimensional hypercube. The t-dimensional hypercube graph (t-hypercube for short), written as B^t, is a t-regular graph with vertex set

$$V(B^t) = \{(x_1, x_2, \cdots, x_t); x_i \in B = \{0,1\}\} \tag{13}$$

and vertices X_1 and X_2 of B^t are adjacent if and only if

$$d_H\left(X_1, X_2\right)=1 \tag{14}$$

where $d_H\left(X_1, X_2\right)$ is the Hamming distance between X_1 and X_2, *i.e.* the number of pairs of different elements of two vectors.

Clearly, the following Eq. (15) holds.

$$\left|E\left(B^t\right)\right|=t \cdot 2^{t-1} \tag{15}$$

Theorem 10 Suppose G is a 4-colorable maximal planar graph with $\delta(G) \geq 4$, f is a 4-coloring of G containing exactly k 2-chromatic cycles, and all these cycles are 2-chromatic unchanged-cycles. Then, $\left|F^f(G)\right|=2^k$, and $G_4^\sigma[F^f(G)]=B^k$.

Proof Suppose f is a 4-coloring of G containing exactly k 2-chromatic cycles $C_1, C_2, \cdots, C_k$, and all these cycles are 2-chromatic unchanged-cycles. For each C_i $(1 \leq i \leq k)$, we denote by 1 if we implement a σ-operation respect to C_i, otherwise by 0. Then, we can establish a none-one correspondence between 2-chromatic unchanged-cycles $C_1, C_2, \cdots, C_k$ and a 0-1 sequence of k-length . Because implementing one σ-operation respect to C_i $(1 \leq i \leq k)$ exactly corresponds to a coloring of G, which means that there is one 0-1 sequence of $\{(x_1, x_2, \cdots, x_k): x_i=0,1, i=1,2, \cdots, k\}$ corresponding to a coloring of $F^f(G)$, it follows that the connected component of G_4^σ containing f has at least 2^k colorings.

On the other hand, without loss of generality, we assume that the k-length 0-1 sequence corresponding to f is $(0,0, \cdots, 0)$. Let $f_i^{\prime}(1 \leq i \leq k)$ be the resulting coloring after implementing a σ-operation respect to $C_i(1 \leq i \leq k)$. Then, f induces exactly k complementary colorings. Similarly, we can prove that each coloring $f_i^{\prime}(1 \leq i \leq k)$ induces exactly k complementary colorings. Therefore, we can further prove that each 4-coloring of the 2^k colorings exactly induces k complementary colorings. Notice that each 4-coloring can induce its complementary coloring if and only if the Hamming distance between the two k-length 0-1 sequences corresponding to the two coloring is equal to 1. Hence, the graph formed by this 2^k 0-1 sequences corresponding to the 2^k 4-colorings is a hypercube.

Obviously, these 2^k 4-colorings are closed under σ-operations , *i.e.* they can not induce any other 4-coloring by σ-operations, except these 2^k 4-colorings. This completes the proof of the theorem.

Now, a question naturally arises. Is any 2-chromatic cycle of any 4-coloring breakable? The answer is positive, we will deal with the issue in later articles of this series. We propose it as a conjecture here.

Conjecture 2 Suppose that G is a 4-colorable maximal planar graph with $\delta \geq 4$. Then, any 2-chromatic cycle in G is breakable.

4.3 Circular-cycle-type Kempe equivalent classes

Let G be a 4-colorable maximal planar graph with $\delta(G) \geq 4$, $\mathbb{C} \subseteq C^2(G)$. If the following three conditions hold: (1) $\forall C_1, C_2 \in \mathbb{C}$, C_1 and C_2 are relevant; (2) $|\mathbb{C}| \geq 2$; (3) $\mathbb{C}$ is a set of maximal relevant cycles, *i.e.* $C^2(G) \backslash \mathbb{C}$ contains no 2-chromatic cycle C' relevant to any 2-chromatic cycle of $\mathbb{C}$. Then, we call each 2-chromatic cycle of $\mathbb{C}$ a circular 2-chromatic cycle, and every coloring of $F^f(G)$ a circular-cycle coloring, where $f \in C_4^0(G)$ contains a 2-chromatic cycle $C \in \mathbb{C}$ relevant to some 2-chromatic cycle of $\mathbb{C} \backslash C$, and $\mathbb{C}$ the set of circular 2-chromatic cycles of $F^f(G)$. Furthermore, the set consisting of f and all colorings obtained by implementing σ-operations starting with f respect to any 2-chromatic cycle in $\mathbb{C}$ is referred to as the set of circular 2-chromatic cycles respect to $\mathbb{C}$, denoted by $F_{\mathbb{C}}^f(G)$. If $F^f(G)$ contains no 2-chromatic unchanged-cycle coloring, then we refer to the Kempe equivalent class containing $F^f(G)$ as the circular-cycle-type Kempe equivalent class. If G contains a circular-cycle-type Kempe equivalent class, then we call G a circular-cycle-type maximal planar graph.

Fig. 13 gives two graphs G and H with their 4-coloring f and g, respectively. It is not hard to verify that f is a 2-chromatic unchanged-cycle coloring, and also a circular-cycle coloring. The specific analysis is as follows.

(1) f is a 2-chromatic unchanged-cycle coloring based on the cycle $C_1=v_1 v_2 v_3 v_4 v_1$;

(2) f is a circular-cycle coloring based on the circular 2-chromatic cycles set $\mathbb{C}=\{C_2, C_3, C_4, C_5,$

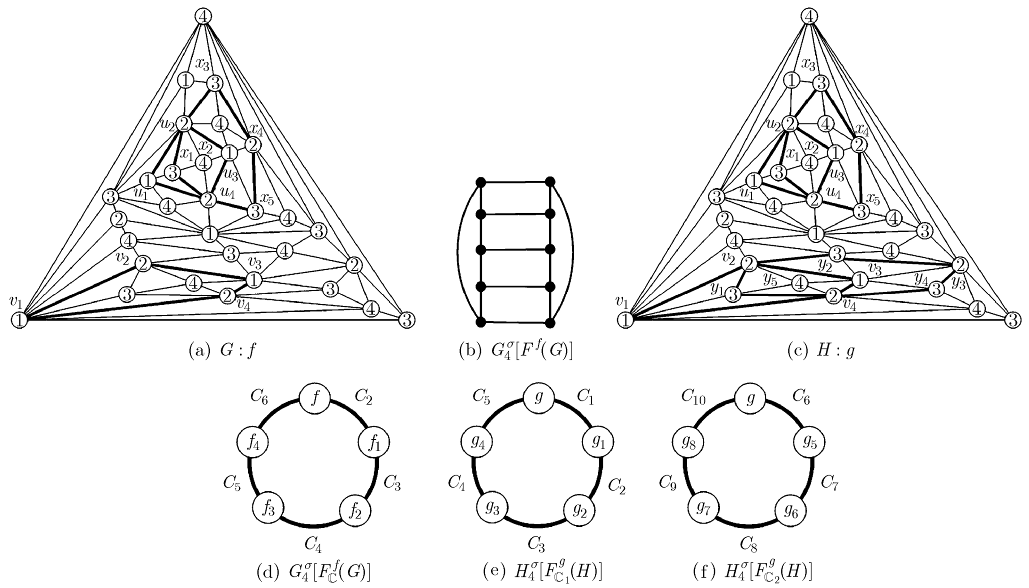


Fig. 13 Two examples of cycle-type and circular-cycle-type Kempe equivalent classes

$C_6\}$, where $C_2 = u_1u_2u_3u_4u_1$, $C_3 = x_1u_2x_3x_4x_5u_4x_1$, $C_4 = x_2u_2x_3x_4x_5u_4x_2$, $C_5 = x_1u_2u_3u_4x_1$, $C_6 = u_1u_2x_2u_4u_1$; the set of circular-cycle colorings respect to $\mathbb{C}$ $F_{\mathbb{C}}^f(G) = \{f, f_1, f_2, f_3, f_4\}$, where $f_1 = \sigma(f, C_2)$, $f_2 = \sigma(f_1, C_3)$, $f_3 = \sigma(f_2, C_4)$, $f_4 = \sigma(f_3, C_5)$. The subgraph of G_4^σ induced by $F_{\mathbb{C}}^f(G)$ is shown in Fig. 13(d).

The coloring g is a 4-coloring of H, and contains two sets of circular 2-chromatic cycles: $\mathbb{C}_1$ and $\mathbb{C}_2$, where $\mathbb{C}_1 = \{C_1, C_2, C_3, C_4, C_5\}$, $C' = f'$. Here, $C_1 = v_1v_2v_3v_4$, $C_2 = y_5v_2y_2y_3y_4v_4$, $C_3 = y_1v_2v_3v_4$, $C_4 = v_1v_2y_5v_4$, $C_5 = y_1v_2y_2y_3y_4v_4$, $C_6 = u_1u_2u_3u_4$, $C_7 = x_2u_2x_3x_4x_5u_4$, $C_8 = x_1u_2u_3u_4$, $C_9 = u_1u_2x_2u_4$, $C_{10} = x_1u_2x_3x_4x_5u_4$. Two circular-cycle coloring sets respect to $\mathbb{C}_1$ and $\mathbb{C}_2$ are $F_{\mathbb{C}_1}^g(H) = \{g, g_1, g_2, g_3, g_4\}$ and $F_{\mathbb{C}_2}^g(H) = \{g, g_5, g_6, g_7, g_8\}$, respectively, where $g_1 = \sigma(g, C_1)$, $g_2 = \sigma(g_1, C_2)$, $g_3 = \sigma(g_2, C_3)$, $g_4 = \sigma(g_3, C_4)$, $g = \sigma(g_4, C_5)$, $g_5 = \sigma(g, C_6)$, $g_6 = \sigma(g_5, C_7)$, $g_7 = \sigma(g_6, C_8)$, $g_8 = \sigma(g_7, C_9)$, $g = \sigma(g_8, C_{10})$. The Fig. 13(e) and Fig. 13(f) illustrate the subgraphs of H_4^σ induced by $F_{\mathbb{C}_1}^g(H)$ and $F_{\mathbb{C}_2}^g(H)$, respectively.

(3) $F^f(G)$ is a cycle-type Kempe equivalent class. The connected component of G_4^σ containing f is shown in Fig. 13(b). $F^g(H)$ is a circular-cycle-type Kempe equivalent class.

From the two examples shown in Fig. 13, the Kempe equivalent class induced by a 4-coloring f of $C_4^0(G)$ is of one of the following types.

Pure cycle type It contains one or more 2-chromatic unchanged-cycles, as shown in Fig. 11 and Fig. 12.

Mixed type It contains not only 2-chromatic unchanged-cycle but also circular 2-chromatic cycle, see Fig. 13(a).

Pure circular-cycle type It contains one or more circular 2-chromatic cycles. See the coloring g shown in Fig. 13(b) that contains two circular 2-chromatic cycles.

Remark 1 There exists some graph that has the same 2-chromatic cycle under different colorings of the graph, but these colorings belong to different Kempe equivalent classes.

Remark 2 Let G be a maximal planar graph with $\delta \geq 4$. It is possible that $C_4^0(G)$ contains 1 to 3 types of Kempe equivalent classes, and also contains more than one Kempe equivalent classes with the same type. For instance, the icosahedron contains ten tree-type equivalent classes.

As we have known, the σ-operation can not induce a Kempe equivalent class of G from another Kempe equivalent class of G. In order to solve this problem, we put forward two approaches to overcome this problem: breaking-cycle method and breaking-tree method. For this, we need to further study cycle-type maximal planar graphs and circular 2-chromatic cycle-type maximal planar graphs, which will be given in the later paper of this series of articles.

5 Kempe Graphs

If a 4-colorable maximal planar graph G with $\delta \geq 4$ is a Kempe graph, then we can induce all 4-colorings of $C_4^0(G)$ from a given 4-coloring by σ-operations. To characterize such graphs, this section introduces a method to recursively construct Kempe graphs based on the extending domino configuration operations, and proposes two conjectures.

5.1 A conjecture of Kempe graphs

For a 4-colorable maximal planar graph G with $\delta(G) \geq 4$, if G is a non-Kempe graph, then there are three types of Kempe equivalent classes: tree-type, cycle-types, and circular-cycle-type.

Conjecture 3 Let G be a 4-colorable maximal planar graph with $\delta(G) \geq 4$. Then G is a Kempe graph if and only if the Kempe equivalent class of G is not tree-type, cycle-type, or circular-cycle-type.

The Conjecture 3 is relevant to Uniquely Four-Colorable Maximal Planar Graph Conjecture. If the Uniquely Four-Colorable Maximal Planar Graph Conjecture is true, that is, every uniquely four-colorable maximal planar graph is a recursive maximal planar graph[26], then each 4-colorable maximal planar graph G with $\delta(G) \geq 4$ has at least two different 4-colorings. If G is tree-type, then G is not a Kempe graph, because any tree coloring can not induce any other 4-coloring of G by σ-operations.

The Conjecture 3 is also relevant to Conjecture 2. Suppose that there exists a 4-coloring $f \in C_4^0(G)$ with a 2-chromatic unchanged-cycle C. If Conjecture 2 is true, then the cycle C is breakable. Consequently, $\exists f' \in C_4^0(G)$, such that $|f'(C)| \geq 3$. But f and f' are unreachable by the σ-operations. Therefore, G is not a Kempe graph.

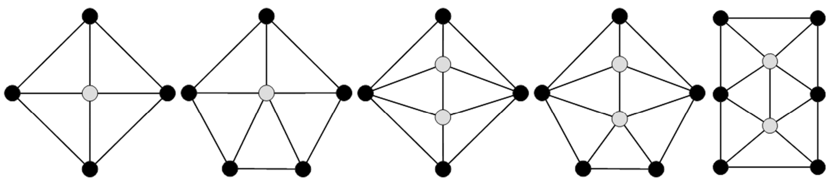
However, even if the Conjecture 3 turns out to be true, we can not know the characteristics of Kempe graphs from its types of Kempe equivalent classes. For further research on Kempe graphs, we then propose the domino recursive construction method.

5.2 Constructions of Kempe graphs

In the second paper of this series of articles[25], we proved that every maximal planar graph G with order $n(\geq 9)$ and $\delta(G) \geq 4$ has an ancestor-graph of order $(n-2)$ or $(n-3)$ and minimum degree not less than 4. In other words, there are at least one of five basic domino configurations of Fig. 14 in G. For convenience of statement, we write these five basic domino configurations shown in Fig. 14 as W_4^1, W_5^1, W_4^2, W_5^2, and W_6^2.

Let G be a 4-colorable maximal planar graph having order $n(\geq 7)$ and $\delta(G) \geq 4$. Suppose that G is non-separable, and P_3 is a path of 2-length. After implementing an extending 4-wheel operation on P_3, there must yield a new domino configuration W_4^1. So we call this operation an extending W_4^1 operation, and the graph obtained by the extending W_4^1 operation is denoted by $\zeta^{W_4^1}(G)$. Similarly, we refer to the processes for obtaining the basic domino configurations shown in Figs. 14(b)~14(e) through the domino extending wheel operations in G as the extending W_5^1 operation, the extending W_4^2 operation, the extending W_5^2 operation, and the extending W_6^2 operation. These basic domino configurations are denoted by $\zeta^{W_5^1}(G)$, $\zeta^{W_4^2}(G)$, $\zeta^{W_5^2}(G)$, and $\zeta^{W_6^2}(G)$, respectively.

Given that the natural coloring f' obtained by implementing a domino extending wheel operation under a 4-coloring f, we can prove Theorem 11 through the following three cases: one of f and f' is tree-type; one of f and f' is cycle-type; one of f and f' is circular cycle-type.



(a) W_4^1　(b) W_5^1　(c) W_4^2　(d) W_5^2　(e) W_6^2

Fig. 14　Five basic domino configurations

Theorem 11 Let G be a 4-colorable maximal planar graph with $\delta(G) \geq 4$. Then $\zeta^{W_4^1}(G)$ and $\zeta^{W_4^2}(G)$ are two Kempe graphs if and only if G is a Kempe graph.

In terms of $\zeta^{W_5^1}(G)$ and $\zeta^{W_5^2}(G)$, we propose a conjecture as follows.

Conjecture 4 Let G be a 4-colorable Kempe maximal planar graph with $\delta(G) \geq 4$. Then $\zeta^{W_5^1}(G)$ and $\zeta^{W_5^2}(G)$ are two Kempe maximal planar graphs if and only if

$$\left|L^4(G)\right| \leq 2 \tag{16}$$

A further research on Kempe maximal planar graphs will be given in later papers of this series of articles, which include the proofs of Theorem 11 and Conjecture 4, as well as the relations between G and $\zeta^{W_5^2}(G)$, *etc.*

6 Conclusion and Prospection

It is generally known that showing the characteristics of Kempe graphs is still a difficult and hot problem. Although there are many literatures in this field, it is hard to find any necessary and sufficient condition of a k-chromatic Kempe graph. Hence, the currently main research focuses on the Kempe equivalence of some special graphs, such as regular graphs. In this series of articles we are concerned with maximal planar graphs.

The main contributions of this paper are summarized as follows: (1) We observe that the inner mechanism that two 4-colorings in maximal planar graphs being Kempe equivalent is closely related with a class of subgraphs, called 2-chromatic ears. So we make an in-depth research on 2-chromatic ears. (2) We introduce and explore the properties of σ-characteristic graphs, which clearly characterize the relations of all 4-colorings of G. (3) We partition the Kempe equivalent classes of non-Kempe graphs into three classes: tree-type, cycle-type, and circular-cycle-type, and point out that all these three classes can exist simultaneously in the set of 4-colorings of one maximal planar graph. (4) In terms of Kempe maximal planar graphs, we make a research on their characteristics, put forward a recursive domino method to construct such graphs, and conjecture that $\zeta^{W_5^1}(G)$ and $\zeta^{W_5^2}(G)$ are two Kempe maximal planar graphs if and only if the number of potential 4-chromatic funnel subgraphs of G is less than two.

We will gradually make comprehensive in-depth studies on three types of Kempe equivalent classes of non-Kempe graphs in later papers of this series of articles. Especially, we will show a necessary and sufficient condition of Kempe maximal planar graphs.

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